Determinants

July 18, 2007

In class we showed that there exists a unique map

\[ D : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}, \quad (v_1, \ldots, v_n) \mapsto D(v_1, \ldots, v_n) \]

satisfying the properties

1. **Multilinear:**
   \[ D(\ldots, \lambda v + \mu w, \ldots) = \lambda D(\ldots, v, \ldots) + \mu D(\ldots, w, \ldots) \]
   for all \( \lambda, \mu \in \mathbb{R} \) and \( v, w \in \mathbb{R}^n \).

2. **Alternating:**
   \[ D(\ldots, v, \ldots, w, \ldots) = -D(\ldots, w, \ldots, v, \ldots) \]
   for all \( v, w \in \mathbb{R}^n \).

3. **Normalized:**
   \[ D(e_1, \ldots, e_n) = 1 \]

We then defined the **determinant** of an \( n \times n \)-matrix \( A \) with column vectors \( a_1, \ldots, a_n \) to be

\[ \det(A) := D(a_1, \ldots, a_n). \quad (1) \]

Therefore the determinant has properties (1)-(3) with respect to the columns of the matrix \( A \).

We proved that for an \( n \times n \)-matrix \( A = (a_{ij}) \) the determinant is given by the explicit formula

\[ \det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1)} 1 \cdots a_{\sigma(n)n}. \quad (2) \]

This formula is sometimes called the **Leibniz formula**, named after the mathematician Gottfried Leibniz (1646-1716).
Problem 1. Use the Leibniz formula to give an explicit formula for the determinant of a 4x4-matrix. (This should be a sum of 24 products, since \(\#S_4 = 24\)).

Solution.

\[
\det(A) = a_{11}a_{22}a_{33}a_{44} + a_{11}a_{32}a_{43}a_{24} + a_{11}a_{42}a_{23}a_{34} \\
+ a_{21}a_{12}a_{43}a_{34} + a_{21}a_{32}a_{13}a_{44} + a_{21}a_{42}a_{33}a_{14} \\
+ a_{31}a_{12}a_{23}a_{44} + a_{31}a_{22}a_{43}a_{14} + a_{31}a_{42}a_{13}a_{24} \\
+ a_{41}a_{12}a_{33}a_{24} + a_{41}a_{22}a_{13}a_{34} + a_{41}a_{32}a_{23}a_{14} \\
- a_{11}a_{22}a_{43}a_{34} - a_{11}a_{32}a_{23}a_{44} - a_{11}a_{42}a_{33}a_{24} \\
- a_{21}a_{12}a_{33}a_{44} - a_{21}a_{32}a_{43}a_{14} - a_{21}a_{42}a_{13}a_{34} \\
- a_{31}a_{12}a_{43}a_{24} - a_{31}a_{22}a_{13}a_{44} - a_{31}a_{42}a_{23}a_{14} \\
- a_{41}a_{12}a_{23}a_{34} - a_{41}a_{22}a_{33}a_{14} - a_{41}a_{32}a_{13}a_{24}
\]

Problem 2. An \(n\times n\)-matrix \(A = (a_{ij})\) is called diagonal if \(a_{ij} = 0\) for \(i \neq j\). Compute the determinant of a diagonal matrix in two different ways. First use the Leibniz formula. Secondly, use the definition (1) and properties (1)-(3).

Solution. In the Leibniz formula the only product which does not involve a zero entry of the matrix \(A\) is the one corresponding to the identity permutation: \(a_{11}a_{22} \ldots a_{nn}\). This proves the claim.

For the second proof let \(a_1, \ldots, a_n\) denote the column vectors of \(A\).

\[
\det(A) = D(a_2, \ldots, a_n) \\
= D(a_{11}e_1, \ldots, a_{nn}e_n) \\
= a_{11} \ldots a_{nn}D(e_1, \ldots, e_n) \quad \text{by multilinearity} \\
= a_{11} \ldots a_{nn} \quad \text{by normalization}
\]

Problem 3. An \(n\times n\)-matrix \(A = (a_{ij})\) is called upper triangular if \(a_{ij} = 0\) for \(i > j\). Show that the determinant of an upper triangular matrix is given by the product of the diagonal entries. Hint: Use the Leibniz formula and realize that only one permutation contributes a nonzero summand.

Solution. Same proof as above, the only permutation which leads to a nonzero product is the identity permutation.

Problem 4. Using properties (1)-(3) show that the determinant of a matrix does not change if we add a multiple of one column to another column.
Solution. Denote the columns of $A$ by $a_1, \ldots, a_n$. Let’s say we add $\lambda a_i$ to the column $a_j$:

\[
D(a_1, \ldots, a_i, \ldots, a_j + \lambda a_i, \ldots, a_n) = D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n)
+ \lambda D(\overbrace{a_1, \ldots, a_i, \ldots, a_i, \ldots, a_n}^{0})
\]

\[
= D(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = \det(A)
\]