THE DISTRIBUTION OF $\mathbb{F}_q$-POINTS ON CYCLIC ℓ-COVERS OF GENUS g

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Abstract. We study the fluctuations in the number of points of ℓ-cyclic covers over the finite field $\mathbb{F}_q$, when q is fixed and the genus tends to infinity. The distribution is given in terms of a sum of $q+1$ i.i.d. random variables. This was completely settled for hyperelliptic curves by Kurlberg and Rudnick [KR09], while statistics were obtained for certain components of the moduli space of ℓ-cyclic covers in [BDFL10]. In this paper, we obtain statistics for the distribution of the number of points as the covers vary over the full moduli space of ℓ-cyclic covers of genus g. This is achieved by relating ℓ-covers to cyclic function field extensions, and counting such extensions with prescribed ramification and splitting conditions at a finite number of primes.

Keywords: curves over finite fields, distribution of number of points, function field extensions, local behavior

Contents

1. Introduction and results

Let q be a prime power, and let $\mathbb{F}_q$ be the finite field with q elements. The goal of this paper is to establish statistics for the distribution of the number of $\mathbb{F}_q$-points of ℓ-cyclic covers C of $\mathbb{P}^1$ defined over $\mathbb{F}_q$, as C varies over the moduli space $\mathcal{H}_{g,\ell}$ of such covers of

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genus g for large g (and fixed q). We always suppose that ℓ is a prime number such that $q \equiv 1 \pmod{\ell}$. For $\ell = 2$ (the case of hyperelliptic curves), this was addressed by Kurlberg and Rudnick [KR09] who showed that the probability that $\#C(\mathbb{F}_q) = m$ for some integer m is the probability that the sum of $q+1$ independent and identically distributed (i.i.d.) random variables is equal to m. This was generalized to cyclic ℓ-covers of degree d by the first, second, third and fifth named authors in [BDFL10] who obtained statistics for each irreducible component of the moduli space

$$\mathcal{H}_{g,\ell} = \bigcup_{\substack{d_1+2d_2+\cdots+(\ell-1)d_{\ell-1}\equiv 0 \pmod{\ell},\\ 2g=(\ell-1)(d_1+\cdots+d_{\ell-1}-2)}} \mathcal{H}^{(d_1,\ldots,d_{\ell-1})}, \tag{1}$$

as $d_1, d_2, \ldots, d_\ell$ tend to infinity. Again, the probability that $\#C(\mathbb{F}_q) = m$ for some integer m, as C varies over $\mathcal{H}^{(d_1,\ldots,d_{\ell-1})}$ and $d_1, \ldots, d_{\ell-1} \to \infty$, is the probability that the sum of $q+1$ i.i.d. random variables is equal to m. The i.i.d. random variables $X_1, \ldots, X_{q+1}$ are given by (for any prime $\ell \geq 2$)

$$X_i = \begin{cases} 0 & \text{with probability } \dfrac{(\ell-1)q}{\ell(q+\ell-1)}, \\[2ex] 1 & \text{with probability } \dfrac{\ell-1}{q+\ell-1}, \\[2ex] \ell & \text{with probability } \dfrac{q}{\ell(q+\ell-1)}. \end{cases} \tag{2}$$

As the statistics hold for $d_1, \ldots, d_{\ell-1} \to \infty$, this does not give statistics for the distribution of the number of points on $\mathcal{H}_{g,\ell}$, since $g \to \infty$ does not mean that $d_1, \ldots, d_{\ell-1} \to \infty$ on all components $\mathcal{H}^{(d_1,\ldots,d_{\ell-1})}$ for a given genus in (1). Other statistics for cyclic ℓ-covers were also obtained by counting the covers in a different way (which does not preserve the genus) by Xiong [Xio10] and Cheong, Wood and Zaman [CWZar], and the distribution of the number of (affine) $\mathbb{F}_q$-points on those covers was also given by a sum of i.i.d. random variables but with different probabilities than the random variables of [BDFL10].

We show in this paper that the statistics for the distribution of the number of $\mathbb{F}_q$-points for covers in $\mathcal{H}_{g,\ell}$ are also given by the random variables (2). The strategy is completely different from the work in [BDFL10]. In this paper we study the equivalent question of counting the number of extensions of the function field $K = \mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, conductor of degree $\mathfrak{n}$, and prescribed splitting/ramification conditions at a finite set of fixed primes of $\mathbb{F}_q(X)$. We explain in Section 5 why these two questions are equivalent, and give general formulas for the number of points on covers in terms of the distribution of their function field extensions.

In order to count the cyclic function field extensions associated to our statistics for point counting on covers, we use a classical approach described in [Wri89] (and first due to Cohn [Coh54] for the case of cubic extensions of the rationals) which is to study the Dirichlet series

$$\sum_{\mathrm{Gal}(L/K)\cong G} \mathfrak{D}(L/K)^{-s},$$

where $\mathfrak{D}(L/K)$ is the absolute norm of the discriminant $\mathrm{Disc}(L/K)$. The approach uses class field theory to give an explicit expression for the Dirichlet series. This is done in generality by Wright in [Wri89] for any global field K and any abelian group G. The count is then obtained by an application of the Tauberian theorem, and the main term is given by the rightmost pole of the Dirichlet series. The order of this pole varies according to the group G and the ground field K (more precisely with the number of roots of unity in K). This is described in [Wri89, Theorem 1.1].

In this paper, we apply those techniques to the case $K = \mathbb{F}_q(X)$ and $G = \mathbb{Z}/\ell\mathbb{Z}$, and we further restrict to counting extensions with prescribed splitting conditions at the $\mathbb{F}_q$-rational places of K. To find our desired statistics for point counts of curves, we need to obtain explicit constants in our asymptotics, and in particular understand how those constants change as we change the splitting conditions. For this, we use the last author's further development of Wright's method in [Woo10], which determines probabilities of various splitting types in abelian extensions of number fields. We are also interested in the secondary terms, and which power saving can be obtained after taking into consideration all secondary terms. Our results can then be used to get the distribution of the number of points on $\mathcal{H}_{g,\ell}$, but also have other applications for statistics on the moduli spaces of curves over finite fields, such as the power of traces and the one-level density, as it is shown in a forthcoming paper of the first and second named author and some collaborators [BCD+14]. We also compute the values of the constants for the leading term of the asymptotic, so the counts obtained with those techniques can be compared with the counts of [BDFL10] (see Section 5.1).

We now state the main results of our paper. We first define some notation. Let $\mathcal{V}_K$ be the set of places of K. Let $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n})$ be the number of extensions of $K = \mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the degree of the conductor is equal to $\mathfrak{n}$. Let $\mathcal{V}_R$, $\mathcal{V}_S$, $\mathcal{V}_I$ denote three finite and disjoint sets of places of $\mathbb{F}_q(X)$, let $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)$ be the number of extensions of $\mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, which are ramified at the places of $\mathcal{V}_R$, split at the places of $\mathcal{V}_S$, and inert at the places of $\mathcal{V}_I$, and such that the degree of the conductor is equal to $\mathfrak{n}$.

Theorem 1.1. *Let $\ell \geq 2$ be a prime, and let $\mathcal{V}_R$, $\mathcal{V}_S$, $\mathcal{V}_I$ and $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)$ be as defined above and let $\mathcal{V} = \mathcal{V}_R \cup \mathcal{V}_S \cup \mathcal{V}_I$. Then,*

$$\begin{aligned} N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}) &= C_\ell \, q^{\mathfrak{n}} P(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right), \\ N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I) &= C_\ell \left(\prod_{v \in \mathcal{V}} c_v\right) q^{\mathfrak{n}} P_{\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I}(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right), \end{aligned}$$

where $P(X), P_{\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I}(X) \in \mathbb{R}[X]$ are monic polynomials of degree $\ell - 2$. Furthermore, C_ℓ is the non-zero constant given by

$$C_\ell = \frac{(1-q^{-2})^{\ell-1}}{(\ell-2)!} \prod_{j=1}^{\ell-2} \prod_{v \in \mathcal{V}_K} \left(1 - \frac{j q^{-2\deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right), \tag{3}$$

and for each place $v \in \mathcal{V}$, we have

$$c_v = \begin{cases} \dfrac{(\ell-1)q^{-\deg v}}{1+(\ell-1)q^{-\deg v}} & \textit{if } v \in \mathcal{V}_R, \\[2ex] \dfrac{1}{\ell(1+(\ell-1)q^{-\deg v})} & \textit{if } v \in \mathcal{V}_S, \\[2ex] \dfrac{\ell-1}{\ell(1+(\ell-1)q^{-\deg v})} & \textit{if } v \in \mathcal{V}_I. \end{cases}$$

Furthermore, for $\ell = 2$ we get the exact count

$$N(\mathbb{Z}/2\mathbb{Z}, \mathfrak{n}) = \begin{cases} 2(q^{\mathfrak{n}} - q^{\mathfrak{n}-2}) & \mathfrak{n} > 2, \mathfrak{n} \textit{ even}, \\ 2q^2 & \mathfrak{n} = 2, \\ 0 & \mathfrak{n} \textit{ odd}. \end{cases}$$

We prove Theorem 1.1 by using class field theory to show that counting $\mathbb{Z}/\ell\mathbb{Z}$ extensions of $\mathbb{F}_q(X)$ is equivalent to counting continuous homomorphisms of the idèle class group of $\mathbb{F}_q(X)$ to $\mathbb{Z}/\ell\mathbb{Z}$. This is the method carried out by [Wri89] for general abelian extensions over function fields and number fields, and also in some recent work of Wood [Woo10] that finds probabilities of various splitting types in abelian extensions of number fields. The idea of obtaining statistics for the families of curves over finite fields by considering the family of function field extensions attached to those curves was also used by Wood in [Woo12] for the family of cyclic trigonal curves (corresponding to non-Galois cubic extensions of $\mathbb{F}_q(X)$), and by Thorne and Xiong [TX14].

We record below a special case of this result which will be needed in some applications to the one-level density in a forthcoming paper [BCD+14]. Then, one needs to study the number of $\mathbb{F}_{q^n}$-points of families of $\mathbb{F}_q$-curves for large n, which can also be done using the results of this paper. The key point is the explicit dependence of each of the coefficients of the polynomial $P_{\mathcal{V},\mathcal{E}}(X)$ with respect to the splitting/ramification conditions to ensure enough cancellation in the ratio of the densities for the split and inert primes. More corollaries of that type can be extracted from the proof of Theorem 1.1 if needed for other applications.

Corollary 1.2. *Let $v \in \mathcal{V}_K$ be a place, let $\epsilon \in \{$ramified, split, inert$\}$, and let $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v, \epsilon)$ be the number of extensions of $\mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the degree of the conductor is equal to $\mathfrak{n}$ and with the prescribed behavior ϵ at the place v. Then,*

$$\begin{aligned} N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v, \textit{ramified}) &= \frac{(\ell-1)q^{-\deg v}}{1+(\ell-1)q^{-\deg v}} C_\ell q^{\mathfrak{n}} P_R(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right) \\ N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v, \textit{split}) &= \frac{1}{\ell(1+(\ell-1)q^{-\deg v})} C_\ell q^{\mathfrak{n}} P_S(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right) \\ N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v, \textit{inert}) &= \frac{1}{\ell(1+(\ell-1)q^{-\deg v})} C_\ell q^{\mathfrak{n}} P_I(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right), \end{aligned}$$

where C_ℓ is the non-zero constant defined by (3), $P_R(X)$ and $P_S(X) \in \mathbb{R}[X]$ are monic polynomials polynomial of degree $\ell - 2$ and $P_I(X) = (\ell-1)P_S(X)$.

Finally, we state the result for the distribution of points on ℓ-cyclic covers of $\mathbb{P}^1$ of fixed genus that can be obtained by Theorem 1.1. This distribution is given in terms of the same random variables from [BDFL10].

Theorem 1.3. *Let $\mathcal{H}_{g,\ell}$ be the moduli space of $\mathbb{Z}/\ell\mathbb{Z}$ Galois covers of $\mathbb{P}^1$ of genus g. Then, as $g \to \infty$,*

$$\frac{|\{C \in \mathcal{H}_{g,\ell}(\mathbb{F}_q) : \#C(\mathbb{F}_q) = m\}|}{|\mathcal{H}_{g,\ell}(\mathbb{F}_q)|} = \operatorname{Prob}\left(X_1 + \dots X_{q+1} = m\right) + O_\ell\left(\frac{1}{g}\right),$$

where the X_i's are independent identically distributed random variables with

$$X_i = \begin{cases} 0 & \text{with probability } \dfrac{(\ell-1)q}{\ell(q+\ell-1)}, \\ 1 & \text{with probability } \dfrac{\ell-1}{q+\ell-1}, \\ \ell & \text{with probability } \dfrac{q}{\ell(q+\ell-1)}. \end{cases}$$

1.1. **Outline of the paper.** In Section 2, we set up notation and use class field theory to translate the counting of extensions to the counting of maps of the idèle class group. We also prove a general form of the Tauberian theorem over function fields that we will need to analyze the Dirichlet series for cyclic extension of $\mathbb{F}_q(X)$ which is a slight generalization of a result in [Ros02]. In Section 3, we define Dirichlet characters over $\mathbb{F}_q(X)$, and we prove analytic properties of some Dirichlet series that will appear in future sections. In Section 4, we prove our main result, Theorem 1.1. In Subsection 4.1, we look at the particular case of $\ell = 2$ where we can get exact results. Finally, we explain in Section 5 how to obtain statistics for the point counting over the moduli space of cyclic ℓ-covers, and we compare our results with those of [BDFL10].

2. Background and setup

In this section, we set up notation and recall basic facts from Galois theory and class field theory that allow us to rephrase our problem in terms of counting continuous homomorphisms from the idèle class group of a function field to a cyclic group of prime order.

Fix ℓ a prime. Throughout the paper $\mathbb{F}_q$ denotes a finite field with $q \equiv 1 \pmod{\ell}$ elements and $K = \mathbb{F}_q(X)$ is the rational function field over $\mathbb{F}_q$.

2.1. **Notation.** We will denote by G_K the absolute Galois group of K, that is the Galois group $\operatorname{Gal}(K^{\mathrm{sep}}/K)$ of the separable closure of K. Let $\mathcal{D}_K^+$ be the set of effective divisors of K. For each place v of K we will use the standard notations K_v for the completion at v, $\mathcal{O}_v$ for the local ring, κ_v for the residue field, and π_v for a uniformizer at v. Recall that the degree of a place v is given by $\deg v = [\kappa_v : \mathbb{F}_q]$ and its norm is $Nv = q^{\deg v}$, the number of elements in the residue field κ_v. Of course, for a place v_f associated to an irreducible polynomial $f \in \mathbb{F}_q[X]$, we have that $\deg v = \deg f$. For the place at infinity associated with the uniformizer $\pi_\infty = 1/X$, we have that $\deg v_\infty = 1$.

2.2. **From covers to field extensions.** A $\mathbb{Z}/\ell\mathbb{Z}$ cover is a pair (C,π) where $C \xrightarrow{\pi} \mathbb{P}^1$ is an ℓ-degree cover map defined over K. Each $\mathbb{Z}/\ell\mathbb{Z}$ cover (C,π) together with an isomorphism $\mathbb{Z}/\ell\mathbb{Z} \to \mathrm{Aut}(C/\mathbb{P}^1)$ corresponds to a Galois extension L of $K = \mathbb{F}_q(X)$ together with a distinguished isomorphism $\tau : \mathrm{Gal}(L/K) \xrightarrow{\tau} \mathbb{Z}/\ell\mathbb{Z}$. We refer to such extensions as ℓ-cyclic extensions. The genus of the curve C relates to the discriminant $\mathrm{Disc}(L/K)$ via the Riemann-Hurwitz formula (see for instance [Ros02, Theorem 7.16]),

$$2g_C - 2 = \ell(2g_{\mathbb{P}^1} - 2) + \deg \mathrm{Disc}(L/K).$$

Since $q \equiv 1 \pmod{\ell}$, there is no wild ramification and each place v of K either ramifies completely, splits completely or is inert. Thus

$$\mathrm{Disc}(L/K) = \sum_{v \text{ ramified in } L} (\ell - 1)v \tag{4}$$

and

$$2g_C = (\ell - 1)\left[-2 + \sum_{v \text{ ramified in } L} \deg v\right],$$

where the sum is taken over the places v of K that ramify in L.

2.3. **From field extensions to maps.** Our translation from counting extensions to counting maps has two steps. First, by Galois theory, ℓ-cyclic extensions L/K with a distinguished isomorphism $\tau : \mathrm{Gal}(L/K) \xrightarrow{\tau} \mathbb{Z}/\ell\mathbb{Z}$ are in one-to-one correspondence with the surjective continuous homomorphisms $G_K \to \mathbb{Z}/\ell\mathbb{Z}$ from the absolute Galois group of K to $\mathbb{Z}/\ell\mathbb{Z}$. By class field theory, the maps $G_K \to \mathbb{Z}/\ell\mathbb{Z}$ are in one-to-one correspondence with the maps $\mathbf{J}_K/K^\times \to \mathbb{Z}/\ell\mathbb{Z}$ from the idèle class group of K to $\mathbb{Z}/\ell\mathbb{Z}$.

Thus an ℓ-cyclic extension L/K of given discriminant corresponds to a nontrivial continuous homomorphism $\varphi : \mathbf{J}_K/K^\times \to \mathbb{Z}/\ell\mathbb{Z}$. We first remark that it suffices to count the maps

$$\phi : \pi_\infty^{\mathbb{Z}} \times \prod_v \mathcal{O}_v^\times \to \mathbb{Z}/\ell\mathbb{Z} \tag{5}$$

which are trivial on $\mathbb{F}_q^\times$, since any such map has a unique extension to $\mathbf{J}_K/K^\times \to \mathbb{Z}/\ell\mathbb{Z}$. (Here $\pi_\infty^{\mathbb{Z}}$ is the free abelian group generated by π_∞.)

There are $\ell - 1$ unramified surjective continuous homomorphisms $\mathbf{J}_K/K^\times \to \mathbb{Z}/\ell\mathbb{Z}$ (one for each generator of $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to the extensions $K(\sqrt[\ell]{\beta}), \ldots, K(\sqrt[\ell]{\beta^{\ell-1}})$). There is also the trivial map, that is also unramified everywhere. In terms of extensions, this corresponds to a K-algebra. In terms of covers of $\mathbb{P}^1$, this corresponds to the split cover that consists of ℓ disjoint copies of $\mathbb{P}^1$.

For each place v of K, the component $\phi_v : \mathcal{O}_v^\times \to \mathbb{Z}/\ell\mathbb{Z}$ is the composition of φ with a canonical map $\mathcal{O}_v^\times \to \mathbf{J}_K \to \mathbf{J}_K/K^\times$. A place v ramifies in L if and only if the map ϕ_v is not trivial on $\mathcal{O}_v^\times$. Let ψ_∞ be the restriction of ϕ to $\pi_\infty^{\mathbb{Z}}$.

Thus the conductor of the map ϕ is

$$\mathrm{Cond}(\phi) = \sum_{v \text{ ramified in } L} v,$$

which is also the conductor of the extension L/K. As there is no wild ramification, the discriminant-conductor formula (see for instance [VS06, Section 12.6]) yields

$$\mathrm{Disc}(L/K) = (\ell - 1)\,\mathrm{Cond}(L/K) = (\ell - 1)\,\mathrm{Cond}(\phi). \tag{6}$$

We now address the global compatibility condition. Fix $\mu \in \mathbb{F}_q$, a generator of the multiplicative group $\mathbb{F}_q^\times$. For each place v of K, we note that the map $\phi_v : \mathcal{O}_v^\times \to \mathbb{Z}/\ell\mathbb{Z}$ factors through $(\mathcal{O}_v/(\pi_v))^\times$. Recall that $\deg v = [\mathcal{O}_v/(\pi_v) : \mathbb{F}_q]$ and thus

$$\mathcal{O}_v/(\pi_v) \cong \mathbb{F}_{q^{\deg v}}.$$

For each v, fix a choice of $g_v \in \mathcal{O}_v$ whose image generates $(\mathcal{O}_v/(\pi_v))^\times \cong (\mathbb{F}_{q^{\deg v}})^\times$ and such that

$$\mu = g_v^{\frac{q^{\deg v}-1}{q-1}}.$$

Hence

$$\begin{aligned}
\phi(\mu) :&= \phi(1, \mu, \mu, \dots) \\
&= \phi((1, \mu, 1, 1, \dots)(1, 1, \mu, 1, \dots)\cdots) \\
&= \phi(1, \mu, 1, 1, \dots) + \phi(1, 1, \mu, 1, \dots) + \dots \\
&= \sum_v \phi_v(\mu).
\end{aligned}$$

This implies

$$\phi(\mu) = \sum_v \phi_v\left(g_v^{\frac{q^{\deg v}-1}{q-1}}\right) = \sum_v \left(\frac{q^{\deg v}-1}{q-1}\right)\phi_v(g_v).$$

We note that $\frac{q^{\deg v}-1}{q-1} = q^{\deg v - 1} + q^{\deg v - 2} + \dots + q + 1 \equiv \deg v \pmod{\ell}$ since $q \equiv 1 \pmod{\ell}$. Thus to have $\mathbb{F}_q^\times$ map to zero it is necessary and sufficient that

$$\sum_{v \in \mathrm{Cond}(\phi)} \phi_v(g_v)\,\deg v \equiv 0 \pmod{\ell}. \tag{7}$$

In order to count the extensions L/K with prescribed splitting/ramification conditions at places v of $K = \mathbb{F}_q(X)$, we have to count the maps ϕ as in (5) satisfying the global compatibility condition (7) with corresponding conditions at places v of $\mathbb{F}_q(X)$, which we describe below.

If v is unramified, we need to distinguish between inert and completely split. Since v is unramified, the map φ_v is trivial on $\mathcal{O}_v^\times$ and therefore its image is dictated by $\varphi_v(\pi_v^{\mathbb{Z}})$. Since this is a subgroup of a simple abelian group, we have only two possibilities: either φ_v is surjective, which corresponds to v being inert; or φ_v is trivial, which corresponds to v being completely split.

If $v = v_\infty$ we can read the splitting behavior from $\phi(\pi_\infty, 1, 1, \dots)$. Namely, we have that $v_\infty \notin \mathrm{Cond}(\phi)$ if and only if $\phi_{v_\infty}(\mathcal{O}_{v_\infty}^\times) = 0$. Therefore:

- v_∞ splits completely in L when $\phi_{v_\infty}(\mathcal{O}_{v_\infty}^\times) = 0$ and $\psi_\infty(\pi_\infty) = 0$,
- v_∞ is inert when $\phi_{v_\infty}(\mathcal{O}_{v_\infty}^\times) = 0$ and $\psi_\infty(\pi_\infty) \neq 0$.

Let $v_0 \neq v_\infty$ be unramified. We denote it as an element in the idèles by putting the infinite component first and the v_0 component second. Then since Frob_{v_0} corresponds to $(1, \pi_{v_0}, 1, 1, \dots)$ under the correspondence from class field theory, v_0 splits if and only if $\varphi(1, \pi_{v_0}, 1, 1, \dots) = 0$. By abuse of notation we also let v_0 represent the monic irreducible polynomial in $K^\times$ corresponding to v_0 and we choose $\pi_{v_0} = v_0$. Thus $\mathrm{val}_{v_0}(v_0) = 1, \mathrm{val}_\infty(v_0) = -\deg v_0$ and $\mathrm{val}_v(v_0) = 0$ for $v \neq v_0, v_\infty$. Since φ is trivial on $K^\times$, we know that

$$\begin{aligned} 0 =& \varphi(v_0, v_0, \dots) = \varphi(v_0, 1, \dots) + \varphi(1, v_0, 1, \dots) + \varphi(1, 1, v_0, 1, \dots) + \dots \\ =& \varphi(\pi_\infty^{-\deg v_0}, 1, \dots) + \varphi(v_0 \pi_\infty^{\deg v_0}, 1, \dots) + \varphi(1, v_0, 1, \dots) + \\ &+ \varphi(1, 1, v_0, 1, \dots) + \varphi(1, 1, 1, v_0, 1, \dots) + \dots. \end{aligned}$$

Since v_0 is monic and $\mathrm{val}_{v_\infty}(v_0 \pi_\infty^{\deg v_0}) = 0$, we have that $\varphi(v_0 \pi_\infty^{\deg v_0}, 1, \dots) = 0$. Denoting by $\varphi_{v_0}(\pi_{v_0})$ the term $\varphi(1, 1, \dots, 1, v_0, 1, \dots, 1)$ where the v_0 is in the v_0 place, we obtain,

$$\psi_\infty(\pi_\infty^{-\deg v_0}) + \varphi_{v_0}(\pi_{v_0}) + \sum_{v \neq v_0, v_\infty} \phi_v(v_0) = 0.$$

Since v_0 splits if and only if $\varphi_{v_0}(\pi_{v_0}) = 0$, we see that:

- v_0 splits if and only if $\phi_{v_0}(\mathcal{O}_{v_0}^\times) = 0$ and

$$\psi_\infty(\pi_\infty^{-\deg v_0}) + \sum_{v \neq v_0, v_\infty} \phi_v(v_0) = 0. \tag{8}$$

- v_0 is inert if and only if $\phi_{v_0}(\mathcal{O}_{v_0}^\times) = 0$ and

$$\psi_\infty(\pi_\infty^{-\deg v_0}) + \sum_{v \neq v_0, v_\infty} \phi_v(v_0) \neq 0.$$

2.4. **Generating series and the Tauberian Theorem.** As in previous work, our strategy is to make use of the Tauberian theorem to deduce an asymptotic formula for the number of field extensions L/K with discriminant of degree $\mathfrak{n}$ from the analytic properties of the generating series

$$\sum_{\mathrm{Gal}(L/K) \cong \mathbb{Z}/\ell\mathbb{Z}} \mathfrak{D}(L/K)^{-s},$$

where $\mathfrak{D}(L/K)$ is the norm of the discriminant $\mathrm{Disc}(L/K)$. As mentioned above, since we are dealing with cyclic extension of prime degree ℓ, the conductor-discriminant relation gives

$$\mathrm{Disc}(L/K) = (\ell - 1)\,\mathrm{Cond}(L/K) \iff \mathfrak{D}(L/K) = N\left(\mathrm{Cond}(L/K)\right)^{\ell-1},$$

and it is more natural to write the generating series as

$$\sum_{\mathrm{Gal}(L/K) \cong \mathbb{Z}/\ell\mathbb{Z}} \mathfrak{D}(L/K)^{-s} := \sum_{f \in \mathcal{D}_K^+} \frac{a_\ell(f)}{Nf^{(\ell-1)s}},$$

where $a_\ell(f)$ is the number of cyclic extensions of degree ℓ of $K = \mathbb{F}_q(X)$ with conductor f. We will then extend this analysis to study the extensions L that are counted by $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)$ as defined in Section 1 by understanding the generating series

$$\sideset{}{'}\sum_{\substack{\mathrm{Gal}(L/K) \cong \mathbb{Z}/\ell\mathbb{Z} \\ \mathcal{V}, \mathcal{E}}} \mathfrak{D}(L/K)^{-s},$$

where the sum now runs over the cyclic extensions of degree ℓ that satisfy all of prescribed the splitting/ramification conditions. Again, we will write this Dirichlet series as

$$\sideset{}{'}\sum_{\substack{\mathrm{Gal}(L/K)\cong\mathbb{Z}/\ell\mathbb{Z}\\ \mathcal{V},\mathcal{E}}} \mathfrak{D}(L/K)^{-s} := \sum_{f\in\mathcal{D}_K^+} \frac{a_\ell(f,\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)}{Nf^{(\ell-1)s}},$$

where $a_\ell(f,\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)$ is the number of cyclic extensions of degree ℓ of $K=\mathbb{F}_q(X)$ with conductor f that satisfy all of the prescribed splitting/ramification conditions.

We now state and prove the version of the Tauberian theorem needed to analyze the Dirichlet series above. More generally, let k be a positive integer, let $a:\mathcal{D}_K^+\to\mathbb{C}$, and $\mathcal{F}(s)$ be the Dirichlet series

$$\mathcal{F}(s)=\sum_{f\in\mathcal{D}_K^+}\frac{a(f)}{Nf^{ks}}.$$

We want a Tauberian theorem that will allow us to evaluate $\sum_{\deg f=\mathfrak{n}} a(f)$ in the situation when the half-plane of absolute convergence is $\mathrm{Re}(s)>1/k$ for some positive integer k, and the function $\mathcal{F}(s)$ has a finite number of poles (of arbitrary multiplicities) on the line $\mathrm{Re}(s)=1/k$. This is a slight generalization of [Ros02, Theorem 17.1].

Since the function q^{-ks}, and therefore $\mathcal{F}(s)$, are periodic with period $2\pi i/(k\log q)$, nothing is lost by confining our attention to the region

$$B_k=\left\{s\in\mathbb{C}:-\frac{\pi i}{k\log q}\le \mathrm{Im}(s)<\frac{\pi i}{k\log q}\right\}, \tag{9}$$

and we will always suppose that s is confined to the region B_k.

Theorem 2.1. *Let k be a positive integer, and let $0<\delta<1/k$. Let $a:\mathcal{D}_K^+\to\mathbb{C}$, and suppose that the Dirichlet series*

$$\mathcal{F}(s)=\sum_{f\in\mathcal{D}_K^+}\frac{a(f)}{Nf^{ks}}$$

converges absolutely for $\mathrm{Re}(s)>1/k$, and is holomorphic on $\{s\in B_k:\mathrm{Re}(s)\ge\delta\}$ except for a finite number of poles on the line $\mathrm{Re}(s)=1/k$. Let $u=q^{-ks}$ and define $F(u)=\mathcal{F}(s)$. Then,

$$\sum_{\deg f=\mathfrak{n}} a(f) = -\sum_{|u|=q^{-1}} \mathrm{Res}_u \frac{F(u)}{u^{\mathfrak{n}+1}} + O\left(q^{\delta k\mathfrak{n}}M\right),$$

where

$$M=\max_{|u|=q^{-k\delta}}|F(u)|=\max_{\mathrm{Re}(s)=\delta}\mathcal{F}(s).$$

Proof. With the change of variable $u=q^{-ks}$, we have that

$$F(u)=\sum_{\mathfrak{n}=0}^{\infty}\left(\sum_{\deg f=\mathfrak{n}} a(f)\right)u^{\mathfrak{n}},$$

and by hypothesis, $F(u)$ is a meromorphic function on the disk $\{u\in\mathbb{C}:|u|\le q^{-k\delta}\}$, except for finitely many poles with $|u|=1/q$. Let $C_\delta=\{u\in\mathbb{C}:|u|=q^{-k\delta}\}$, oriented counterclockwise. Choose any $\eta>1$ and let $C_\eta=\{u\in\mathbb{C}:|u|=q^{-\eta}\}$, oriented clockwise.

Notice that $\frac{F(u)}{u^{\mathfrak{n}+1}}$ is a meromorphic function between the two circles C_η and C_δ with finitely many poles at $|u| = 1/q$. Thus, by the Cauchy's integral formula,

$$\frac{1}{2\pi i}\oint_{C_\delta + C_\eta} \frac{F(u)}{u^{\mathfrak{n}+1}} du = \sum_{|u|=q^{-1}} \mathrm{Res}_u \frac{F(u)}{u^{\mathfrak{n}+1}}.$$

Since $q^{-\eta} < 1$, using the power series expansion of $F(u)$ around $u = 0$, we have that

$$\frac{1}{2\pi i}\oint_{C_\eta} \frac{F(u)}{u^{\mathfrak{n}+1}} du = -\sum_{\deg f = \mathfrak{n}} a(f).$$

Therefore, we obtain

$$\sum_{\deg f=\mathfrak{n}} a(f) = -\sum_{|u|=q^{-1}} \mathrm{Res}_u \frac{F(u)}{u^{\mathfrak{n}+1}} + \frac{1}{2\pi i}\oint_{C_\delta} \frac{F(u)}{u^{\mathfrak{n}+1}} du.$$

Let M be the maximum of $|F(u)|$ over C_δ. Then

$$\left|\frac{1}{2\pi i}\oint_{C_\delta} \frac{F(u)}{u^{\mathfrak{n}+1}} du\right| \leq M q^{\delta k \mathfrak{n}},$$

which proves the proposition. □

3. Dirichlet characters and L-functions

In this section, we define ℓth-power residue symbols over $\mathbb{F}_q[X]$ (we recall that $q \equiv 1 \pmod{\ell}$). We refer the reader to [Mor91] for details. We then study some auxiliary functions built out of the ℓth-power residue symbols that will be used in the proofs of our main results.

Recall that ℓ is a prime such that $q \equiv 1 \pmod{\ell}$. Thus $\mathbb{F}_q^\times$ contains the ℓth roots of unity. In particular, $b_\ell = \mu^{\frac{q-1}{\ell}}$ is one of these roots where μ is a fixed generator of $\mathbb{F}_q^\times$. Let $v = v(X) \in \mathbb{F}_q[X]$ be a monic irreducible polynomial. We define the ℓ-th power residue symbol as follows. Let

$$\left(\frac{\cdot}{v}\right)_\ell : (\mathbb{F}_q[X]/v(X))^\times \to \mathbb{F}_q^\times$$

be defined by

$$\left(\frac{f}{v}\right)_\ell \equiv f^{\frac{Nv-1}{\ell}} \pmod{v}.$$

In other words, the ℓ-th power residue symbol is given by an ℓ-th root of unity.

Recall that the choice of μ determined for each place v a generator g_v of

$$(\mathcal{O}_v/(\pi_v))^\times \cong (\mathbb{F}_q[X]/(v(X)))^\times \cong (\mathbb{F}_{q^{\deg v}})^\times$$

such that $\mu = g_v^{\frac{q^{\deg v}-1}{q-1}}$. We have

$$g_v^{\frac{q^{\deg v}-1}{\ell}} = \left(g_v^{\frac{q^{\deg v}-1}{q-1}}\right)^{\frac{q-1}{\ell}} = \mu^{\frac{q-1}{\ell}} = b_\ell.$$

By the definition of the ℓ-th power symbol,

$$\left(\frac{g_v}{v}\right)_\ell \equiv b_\ell \pmod{v}.$$

We let σ be an ℓ-order character from $\mathbb{F}_q^\times \to \mathbb{C}^\times$. Then,

$$\chi_{v,\ell} := \sigma \circ \left(\frac{\cdot}{v}\right)_\ell$$

is a Dirichlet character $\chi : \mathbb{F}_q[X] \to \mathbb{C}^\times$ of modulus v, where we define $\chi_{v,\ell}(f(x)) = 0$ if $v(x)$ divides $f(x)$.

For the infinite place v_∞, we further define

$$\chi_{v,\ell}(v_\infty) = \begin{cases} 1 & \deg v \equiv 0 \pmod{\ell}, \\ 0 & \deg v \not\equiv 0 \pmod{\ell}. \end{cases}$$

For χ a Dirichlet character, we denote by $L(s,\chi)$ the Dirichlet L-function

$$L(s,\chi) = \sum_{\substack{F \in \mathbb{F}_q[X] \\ F \text{ monic}}} \frac{\chi(F)}{|F|^s}$$

where F varies over the monic polynomials of $\mathbb{F}_q[X]$, and by $L^*(s,\chi)$ the completed L-function that includes the place at infinity. For a Dirichlet character modulo a monic polynomial v, we have that

$$L^*(s,\chi) = (1-q^{-s})^{-\lambda_v} L(s,\chi),$$

where λ_v is 1 if $\deg v \equiv 0 \pmod{\ell}$, and 0 otherwise.

Then, for χ nontrivial, we remark that both $L(s,\chi)$ and $L^*(s,\chi)$ are analytic and non-zero for $\mathrm{Re}(s) > 1/2$.

By ℓ-power reciprocity, we can write the Kronecker symbol as

$$\chi_{v,\ell}(v_0) = \sigma \circ \left(\frac{v_0}{v}\right)_\ell = \sigma\left(\left((-1)^{(q-1)/\ell}\right)^{\deg v_0 \deg v} \left(\frac{v}{v_0}\right)_\ell\right) = \Psi_{v_0,\ell}(v)\chi_{v_0,\ell}(v), \tag{10}$$

where $\chi_{v_0,\ell}(v)$ is the Dirichlet character modulo v_0 defined above, and $\Psi_{v_0,\ell}(v)$ depends only on the degree of v.

If $v = v_\infty$, let a_n be the principal coefficient of f. Then we define

$$\chi_{v_\infty,\ell}(f) := \begin{cases} \sigma(a_n) & \deg f \equiv 0 \pmod{\ell}, \\ 0 & \deg f \not\equiv 0 \pmod{\ell}. \end{cases}$$

We note that the above definition agrees with ℓ-power reciprocity in the following way

$$\chi_{v,\ell}(v_\infty) = \left((-1)^{(q-1)/\ell}\right)^{\deg v} \chi_{v_\infty,\ell}(v) = \begin{cases} 1 & \deg v \equiv 0 \pmod{\ell}, \\ 0 & \deg v \not\equiv 0 \pmod{\ell}. \end{cases} \tag{11}$$

where we have used that v is a monic polynomial, which implies that $\chi_{v_\infty,\ell}(v) = 1$ when $\ell \mid \deg v$, and that $\left((-1)^{(q-1)/\ell}\right)^{\deg v} = 1$ when $\ell \mid \deg v$ and q odd, and is trivially 1 when q is even since then we have even characteristic.

Finally, we remark that by the above, the Kronecker symbol codifies ramification in extensions in the usual way. Let $f \in \mathbb{F}_q[X]$ (not necessarily monic). Then,

$$\chi_{v,\ell}(f) = \begin{cases} 1 & v \text{ splits in } K(\sqrt[\ell]{f}), \\ \xi_\ell^k, \text{ for some } 1 \le k \le \ell-1 & v \text{ is inert in } K(\sqrt[\ell]{f}), \\ 0 & v \text{ ramifies in } K(\sqrt[\ell]{f}), \end{cases}$$

where ξ_ℓ is a primitive ℓth root of 1.

Lemma 3.1. *Let χ be a nontrivial Dirichlet character and let Ψ be a function on $\mathbb{F}_q[X]$ such that $\Psi(F) = \Psi(G)$ when $\deg F = \deg G$. Then*

$$L(s, \Psi\chi) = \sum_{\substack{F \in \mathbb{F}_q[X] \\ F \text{ monic}}} \frac{\Psi(F)\chi(F)}{|F|^s}$$

is an analytic function on $\mathbb{C}$.

Proof. Let

$$A(n, \Psi, \chi) = \sum_{\substack{F \in \mathbb{F}_q[X], \\ F \text{ monic}, \\ \deg F = n}} \Psi(F)\chi(F).$$

Then

$$L(s, \Psi\chi) = \sum_{n=0}^{\infty} \frac{A(n, \Psi, \chi)}{q^{ns}}.$$

We note that

$$A(n, \Psi, \chi) = \Psi(G) \sum_{\substack{F \in \mathbb{F}_q[X], \\ F \text{ monic}, \\ \deg F = n}} \chi(F)$$

for any polynomial G of degree n, and thus $A(n, \Psi, \chi) = 0$ if n is greater than or equal to the degree of the modulus of χ by the orthogonality relations of characters. $\square$

Lemma 3.2. *Let ξ_ℓ be a primitive ℓth root of 1. Let $\mathcal{V}_R, \mathcal{V}_S$ and $\mathcal{V}_U$ be finite subsets of places of $\mathcal{V}_K$ such that $\mathcal{V}_S = \{v_1, \ldots, v_n\} \subset \mathcal{V}_U$, and $\mathcal{V}_U \cap \mathcal{V}_R = \varnothing$. For each $0 \le j \le \ell - 1$, and each tuple $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$ with $0 \le k_i \le \ell - 1$, let*

$$\begin{aligned} &\mathcal{M}_{j,k_1,\ldots,k_n}(s; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_U) \\ &:= \prod_{v \notin \mathcal{V}_R \cup \mathcal{V}_U} \left(1 + \left(\xi_\ell^{j \deg v} \prod_{h=1}^{n} \chi_{v,\ell}(v_h)^{k_h} + \cdots + \xi_\ell^{(\ell-1) j \deg v} \prod_{h=1}^{n} \chi_{v,\ell}(v_h)^{(\ell-1)k_h}\right) N v^{-(\ell-1)s}\right). \end{aligned}$$

Then, each $\mathcal{M}_{j,k_1,\ldots,k_n}(s; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_U)$ converges absolutely for $\mathrm{Re}(s) > \frac{1}{\ell-1}$ and has analytic continuation to the region $\mathrm{Re}(s) > \frac{1}{2(\ell-1)}$.

In the case where we have only one place $v_0 \in \mathcal{V}_K$ with prescribed ramification $\epsilon_0 \in \{\text{ramified}, \text{split}, \text{inert}\}$, we will denote the above function by

$$\mathcal{M}_{j,k}(s; v_0, \epsilon_0) := \mathcal{M}_{j,k_1}(s; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_U). \tag{12}$$

Proof. For the absolute convergence, we have that the convergence of $\prod_v (1 + (\ell - 1)|Nv^{-s(\ell-1)}|)$ is equivalent to that of $\sum_v \frac{1}{Nv^{s(\ell-1)}}$ and this convergence follows in the same way as the absolute convergence for the zeta function $\zeta_K(s)$ in $\mathrm{Re}(s) > 1$.

For the analytic continuation, we write

$$\begin{aligned}
&\mathcal{M}_{j,k_1,\dots,k_n}(s;\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_U)\\
&= \mathcal{C}^1_{j,k_1,\dots,k_n}(s)\prod_{i=1}^{\ell-1}\prod_{v\notin\mathcal{V}_R\cup\mathcal{V}_U}\left(1+\xi_\ell^{ij\deg v}\prod_{h=1}^{n}\chi_{v,\ell}(v_h)^{ik_h}Nv^{-(\ell-1)s}\right)\\
&= \mathcal{C}^2_{j,k_1,\dots,k_n}(s)\prod_{i=1}^{\ell-1}\prod_{v\notin\mathcal{V}_R\cup\mathcal{V}_U}\left(1-\xi_\ell^{ij\deg v}\prod_{h=1}^{\ell-1}\Psi_{v_h,\ell}(v)^{ik_h}\chi_{v_h,\ell}(v)^{ik_h}Nv^{-(\ell-1)s}\right)^{-1},
\end{aligned}$$

where we have used ℓ-power reciprocity (10), and where $\mathcal{C}^1_{j,k_1,\dots,k_n}(s)$ and $\mathcal{C}^2_{j,k_1,\dots,k_n}(s)$ are analytic functions for $\mathrm{Re}(s)>1/2(\ell-1)$ as the Euler products converge absolutely in that region. For each $1\le i\le\ell-1$, each $0\le j\le\ell-1$ and each tuple $(k_1,\dots,k_n)$ as above, we have that the functions

$$\begin{aligned}
L_{i,j,k_1,\dots k_n}(s) &= \prod_{v\notin\mathcal{V}_R\cup\mathcal{V}_U}\left(1-\xi_\ell^{ij\deg v}\prod_{h=1}^{\ell-1}\Psi_{v_h,\ell}(v)^{ik_h}\chi_{v_h,\ell}(v)^{ik_h}Nv^{-(\ell-1)s}\right)^{-1}\\
&= L(s_1,\Psi_{i,j,k_1,\dots,k_h}\ \chi_{i,j,k_1,\dots,k_h})
\end{aligned}$$

are twisted Dirichlet functions as in Lemma 3.1, where $s_1=(\ell-1)s$,

$$\begin{aligned}
\Psi_{i,j,k_1,\dots,k_h}(v) &= \xi_\ell^{ij\deg v}\prod_{h=1}^{\ell-1}\Psi_{v_h,\ell}(v)^{ik_h}\\
\chi_{i,j,k_1,\dots,k_h}(v) &= \prod_{h=1}^{\ell-1}\chi_{v_h,\ell}(v)^{ik_h}.
\end{aligned}$$

Then, $\Psi_{i,j,k_1,\dots,k_h}(v)$ depends only on the degree of v, and $\chi_{i,j,k_1,\dots,k_h}(v)$ is a non-trivial Dirichlet character since $1\le i\le\ell-1$, and $(k_1,\dots,k_n)\neq0$. Applying Lemma 3.1, this completes the proof of the analytic continuation.

$\square$

Let ξ_ℓ be a primitive ℓth root of 1. We now prove a result bounding the meromorphic continuation of the functions

$$\mathcal{A}(s) := \prod_v\left(1+(\ell-1)Nv^{-(\ell-1)s}\right) \tag{13}$$

$$\mathcal{B}(s) := \prod_v\left(1+(\xi_\ell^{\deg v}+\cdots+\xi_\ell^{(\ell-1)\deg v})Nv^{-(\ell-1)s}\right) \tag{14}$$

on the line $\mathrm{Re}(s)=\frac{1}{2(\ell-1)}+\varepsilon$, for any $\varepsilon>0$. We remark that the Euler products converge (absolutely and uniformly) for $\mathrm{Re}(s)>1/(\ell-1)$.

Lemma 3.3. *Let $0<\varepsilon<\frac{1}{2(\ell-1)}$. The functions $\mathcal{A}(s)$ and $\mathcal{B}(s)$ have meromorphic continuation to the region $\mathrm{Re}(s)>\frac{1}{2(\ell-1)}+\varepsilon$, and their only singularities are poles on the line $\mathrm{Re}(s)=1/(\ell-1)$. Furthermore, both functions are absolutely bounded on the region $\frac{1}{2(\ell-1)}<\mathrm{Re}(s)<\frac{1}{\ell-1}$.*

Proof. For $\mathrm{Re}(s) > \frac{1}{\ell-1}$, we have

$$
\begin{aligned}
\mathcal{A}(s) &= \prod_v \left(1 + (\ell-1) Nv^{-(\ell-1)s}\right) \\
&= \zeta_K((\ell-1)s)^{\ell-1} \prod_v \left(1 + (\ell-1) Nv^{-(\ell-1)s}\right) \left(1 - Nv^{-(\ell-1)s}\right)^{\ell-1} \\
&= \zeta_K((\ell-1)s)^{\ell-1} \prod_v \left(1 + (\ell-1) Nv^{-(\ell-1)s}\right) \Big(1 - (\ell-1) Nv^{-(\ell-1)s} \\
&\qquad + \binom{\ell-1}{2} Nv^{-2(\ell-1)s} + Nv^{-3(\ell-1)s} O_\ell(1)\Big) \\
&= \zeta_K((\ell-1)s)^{\ell-1} \prod_v \left(1 - \binom{\ell-1}{2} Nv^{-2(\ell-1)s} + Nv^{-3(\ell-1)s} O_\ell(1)\right) \\
&= \mathcal{C}(s) \zeta_K((\ell-1)s)^{\ell-1} \prod_v \left(1 - Nv^{-2(\ell-1)s}\right)^{\frac{\ell(\ell-1)}{2}} \\
&= \mathcal{C}(s) \frac{\zeta_K((\ell-1)s)^{\ell-1}}{\zeta_K(2(\ell-1)s)^{\frac{\ell(\ell-1)}{2}}},
\end{aligned}
$$

where $\mathcal{C}(s)$ is analytic for $\mathrm{Re}(s) > \frac{1}{3(\ell-1)} + \varepsilon$. Thus for $s = \frac{1}{2(\ell-1)} + \varepsilon$, as ε goes to zero, the function $\mathcal{A}(s)$ converges to zero, and the result follows. The poles are given by those of $\zeta_K((\ell-1)s)$, namely $s = 1/(\ell-1)$, with multiplicity $\ell-1$.

Similarly, for $\mathrm{Re}(s) > \frac{1}{\ell-1}$, we have

$$
\begin{aligned}
\mathcal{B}(s) &= \prod_v \left(1 + (\xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v}) Nv^{-(\ell-1)s}\right) \\
&= \prod_{j=1}^{\ell-1} Z_K(\xi_\ell^j u) \prod_v \left(1 + (\xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v}) Nv^{-(\ell-1)s}\right) \prod_{j=1}^{\ell-1} \left(1 - \xi_\ell^{j \deg v} Nv^{-(\ell-1)s}\right),
\end{aligned}
$$

where $u = q^{-(\ell-1)s}$ and

$$
Z_K(u) := \frac{1}{(1-qu)(1-u)}
$$

is the zeta function of K.

Thus, we have,

$$\begin{aligned}
\mathcal{B}(s) &= \prod_{j=1}^{\ell-1} Z_K(\xi_\ell^j u) \prod_v \Big(1 + (\xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v}) Nv^{-(\ell-1)s}\Big) \\
&\times \prod_v \Bigg(1 - (\xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v}) Nv^{-(\ell-1)s} \\
&\quad + \Bigg(\sum_{1\le i<j\le \ell-1} \xi_\ell^{i\deg v}\xi_\ell^{j\deg v}\Bigg) Nv^{-2(\ell-1)s} + Nv^{-3(\ell-1)s} O_\ell(1)\Bigg) \\
&= \mathcal{C}(s) \prod_{j=1}^{\ell-1} Z_K(\xi_\ell^j u) \prod_v \left(1 + c(\ell) Nv^{-2(\ell-1)s}\right),
\end{aligned}$$

where

$$\begin{aligned}
c(\ell) &= -\left(\xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v}\right)^2 + \sum_{1\le i<j\le \ell-1} \xi_\ell^{i\deg v}\xi_\ell^{j\deg v} \\
&= -\sum_{1\le i\le j\le \ell-1} \xi_\ell^{i\deg v}\xi_\ell^{j\deg v} \\
&= \begin{cases} -\frac{\ell(\ell-1)}{2} & \ell \mid \deg v, \ell > 2, \\ 0 & \ell \nmid \deg v, \ell > 2, \\ -1 & \ell = 1, \end{cases}
\end{aligned}$$

and $\mathcal{C}(s)$ is analytic for $\mathrm{Re}(s) > \frac{1}{3(\ell-1)} + \varepsilon$. Thus for $s = \frac{1}{2(\ell-1)} + \varepsilon$, as $\varepsilon \to 0$, the function $\mathcal{B}(s)$ converges to 0, and the result follows.

The poles are those of $Z_K(\xi_\ell^j u)$, namely, poles of order one at $s = \frac{1}{\ell-1} + \frac{2j\pi i}{(\ell-1)\ell \log q}$. $\square$

4. ℓ-Cyclic Extensions

In this section, we will give the proofs of the main results of this paper. We will continue with the notation introduced in the earlier sections. Recall that, for a fixed prime ℓ, $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n})$ denotes the number of extensions of K with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the degree of the conductor is $\mathfrak{n}$. As before, ξ_ℓ will always stand for a primitive ℓth root of 1.

Theorem 4.1. *Let $\ell \in \mathbb{Z}$ be a prime. We have*

$$N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}) = C_\ell\, q^{\mathfrak{n}} P_\ell(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right), \tag{15}$$

where $P_\ell(X) \in \mathbb{R}[X]$ is a monic polynomial of degree $\ell - 2$, and where C_ℓ is the non-zero constant given by

$$C_\ell = \frac{(1-q^{-2})^{\ell-1}}{(\ell-2)!} \prod_{j=1}^{\ell-2} \prod_v \left(1 - \frac{jq^{-2\deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right).$$

Proof. To count $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n})$, we consider the Dirichlet series $\mathcal{F}(s)$, which is the generating function with an added constant, namely,

$$\mathcal{F}(s) := \ell + \sum_{\mathrm{Gal}(L/K)\cong\mathbb{Z}/\ell\mathbb{Z}} \mathfrak{D}(L/K)^{-s}.$$

We claim that

$$\begin{aligned}
\mathcal{F}(s) &= \sum_{j=0}^{\ell-1}\prod_v \left(1+(\xi_\ell^{j\deg v}+\cdots+\xi_\ell^{(\ell-1)j\deg v})Nv^{-(\ell-1)s}\right)\\
&= \prod_v \left(1+(\ell-1)Nv^{-(\ell-1)s}\right)+(\ell-1)\prod_v \left(1+(\xi_\ell^{\deg v}+\cdots+\xi_\ell^{(\ell-1)\deg v})Nv^{-(\ell-1)s}\right)\\
&= \mathcal{A}(s)+(\ell-1)\mathcal{B}(s).
\end{aligned}$$

Indeed, we recall from Section 2 that we have to count the maps $\phi : \pi_\infty^{\mathbb{Z}} \times \prod_v \mathcal{O}_v^\times \to \mathbb{Z}/\ell\mathbb{Z}$ satisfying (7). Let $\mathrm{Cond}(\phi)$ be the conductor of such a map ϕ, and v a place of the conductor. In the first line above, the ith term $\xi_\ell^{ij\deg v}Nv^{-(\ell-1)s}$ in each Euler product corresponds to the map where $\phi_v(g_v) = i$ for $1 \le i \le \ell-1$. Therefore, considering all the places v of $\mathrm{Cond}(\phi)$, the term in the jth Dirichlet series above corresponding to the global map ϕ equals

$$\left(\xi_\ell^{\sum_v j\phi_v(g_v)\deg v}\right) \times N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$$

for $0 \le j \le \ell-1$. Thus the sum of those terms over the index j will yield $\ell N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$ if $\sum_v \phi_v(g_v)\deg v \equiv 0 \pmod{\ell}$ and 0 otherwise, and we recover (7). Notice that the ℓ factor multiplying $N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$ is counting the different extensions with the same conductor $K(\sqrt[\ell]{f}), K(\sqrt[\ell]{\beta f}), \dots, K(\sqrt[\ell]{\beta^{\ell-1}f})$ for $\beta \in \mathbb{F}_q^\times$ not an ℓth power. Similarly, the constant ℓ in the definition of $\mathcal{F}(s)$ accounts for the extensions $K(\sqrt[\ell]{\beta}), \dots, K(\sqrt[\ell]{\beta^{\ell-1}})$ for $\beta \in \mathbb{F}_q^\times$ not an ℓth power, as well as the K-algebra given by the completely split cover.

Using the identity

$$\frac{1+(\ell-1)u}{(1+u)^{\ell-1}} = \prod_{j=1}^{\ell-2}\left(1-\frac{ju^2}{(1+u)(1+ju)}\right),$$

we write

$$\begin{aligned}
\mathcal{A}(s) &= \prod_v \left(1+(\ell-1)Nv^{-(\ell-1)s}\right)\\
&= \left(\frac{\zeta_K((\ell-1)s)}{\zeta_K(2(\ell-1)s)}\right)^{\ell-1}\prod_{j=1}^{\ell-2}\prod_v\left(1-\frac{jNv^{-2(\ell-1)s}}{(1+Nv^{-(\ell-1)s})(1+jNv^{-(\ell-1)s})}\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\mathcal{B}(s) &= \prod_v \left(1+(\xi_\ell^{\deg v}+\cdots+\xi_\ell^{(\ell-1)\deg v})Nv^{-(\ell-1)s}\right)\\
&= \prod_v\prod_{j=1}^{\ell-1}\left(1+\xi_\ell^{j\deg v}Nv^{-(\ell-1)s}\right)\prod_v \frac{\left(1+(\xi_\ell^{\deg v}+\cdots+\xi_\ell^{(\ell-1)\deg v})Nv^{-(\ell-1)s}\right)}{\prod_{j=1}^{\ell-1}\left(1+\xi_\ell^{j\deg v}Nv^{-(\ell-1)s}\right)}.
\end{aligned}$$

Recall from Lemma 3.3 that $\mathcal{A}(s)$ is a meromorphic function on $\mathrm{Re}(s) > \frac{1}{2(\ell-1)}$ with a pole of order $\ell-1$ at $s = \frac{1}{\ell-1}$ in the region $B_{\ell-1}$ as defined in (9). The function $\mathcal{B}(s)$ is also meromorphic in $\mathrm{Re}(s) > \frac{1}{2(\ell-1)}$, with simple poles at $s_j = \frac{1}{\ell-1} + \frac{2j\pi i}{(\ell-1)\ell \log q}$ for $|2j| < \ell$ in the region $B_{\ell-1}$.

We set $u = q^{-(\ell-1)s}$, and write $A(u) := \mathcal{A}(s)$ and $B(u) := \mathcal{B}(s)$. Thus,

$$A(u) = \left(\frac{(1-qu^2)(1+u)}{(1-qu)}\right)^{\ell-1} \prod_{j=1}^{\ell-2}\prod_{v}\left(1 - \frac{ju^{2\deg v}}{(1+u^{\deg v})(1+ju^{\deg v})}\right)$$

$$\begin{aligned} B(u) &= \prod_{v}\prod_{j=1}^{\ell-1}\left(1+(\xi_\ell^j u)^{\deg v}\right)\prod_{v}\frac{\left(1+(\xi_\ell^{\deg v}+\cdots+\xi_\ell^{(\ell-1)\deg v})u^{\deg v}\right)}{\prod_{j=1}^{\ell-1}\left(1+(\xi_\ell^j u)^{\deg v}\right)} \\ &= \prod_{j=1}^{\ell-1}\frac{Z_K(\xi_\ell^j u)}{Z_K(\xi_\ell^{2j}u^2)}\prod_{v}\frac{\left(1+b(v)u^{\deg v}\right)}{\prod_{j=1}^{\ell-1}\left(1+(\xi_\ell^j u)^{\deg v}\right)}, \end{aligned}$$

where

$$b(v) = \xi_\ell^{\deg v} + \cdots + \xi_\ell^{(\ell-1)\deg v},$$

and

$$Z_K(u) = \frac{1}{(1-qu)(1-u)}.$$

Fix any δ with $\frac{1}{2(\ell-1)} < \delta < \frac{1}{\ell-1}$. Then $A(u)$ and $B(u)$ are meromorphic functions on the disk $\{u : |u| \le q^{-\delta}\}$. We see that $A(u)$ has a pole of order $\ell-1$ at $u = 1/q$ and $B(u)$ has $(\ell-1)$-many simple poles at $u = (q\xi_\ell^j)^{-1}$ for $j = 1, \cdots, \ell-1$. Then, applying Theorem 2.1 and Lemma 3.3 to $\mathcal{F}(s) = \mathcal{A}(s) + (\ell-1)\mathcal{B}(s)$ with $\delta = \frac{1}{2(\ell-1)} + \varepsilon$ for $\varepsilon > 0$, we have that

$$N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}) = -\mathrm{Res}_{u=q^{-1}}\frac{A(u)}{u^{\mathfrak{n}+1}} - \sum_{j=1}^{\ell-1}\mathrm{Res}_{u=(q\xi_\ell^j)^{-1}}\frac{B(u)}{u^{\mathfrak{n}+1}} + O\left(q^{(1/2+\varepsilon)\mathfrak{n}}\right). \tag{16}$$

We compute,

$$\begin{aligned} &\mathrm{Res}_{u=q^{-1}}\frac{A(u)}{u^{\mathfrak{n}+1}} \\ &= \lim_{u\to q^{-1}}\frac{1}{(\ell-2)!}\frac{d^{\ell-2}}{du^{\ell-2}}(u-q^{-1})^{\ell-1}\frac{1}{u^{\mathfrak{n}+1}}\left(\frac{(1-qu^2)(1+u)}{(1-qu)}\right)^{\ell-1}\prod_{j=1}^{\ell-2}\prod_{v}\left(1-\frac{ju^{2\deg v}}{(1+u^{\deg v})(1+ju^{\deg v})}\right) \\ &= \lim_{u\to q^{-1}}\frac{1}{(\ell-2)!}\frac{d^{\ell-2}}{du^{\ell-2}}\left(\frac{(-(1-qu^2)(1+u))^{\ell-1}}{q^{\ell-1}u^{\mathfrak{n}+1}}\right)\prod_{j=1}^{\ell-2}\prod_{v}\left(1-\frac{ju^{2\deg v}}{(1+u^{\deg v})(1+ju^{\deg v})}\right). \end{aligned}$$

Let

$$H_\ell(u) := \frac{1}{(\ell-2)!}\left(\frac{(-(1-qu^2)(1+u))^{\ell-1}}{q^{\ell-1}}\right)\prod_{j=1}^{\ell-2}\prod_{v}\left(1-\frac{ju^{2\deg v}}{(1+u^{\deg v})(1+ju^{\deg v})}\right). \tag{17}$$

Then, using the product rule for derivatives, we get

$$\begin{aligned}\operatorname{Res}_{u=q^{-1}} \frac{A(u)}{u^{\mathfrak{n}+1}} &= \lim_{u\to q^{-1}} \sum_{i=0}^{\ell-2} \binom{\ell-2}{i} \frac{d^i}{du^i}\left(\frac{1}{u^{\mathfrak{n}+1}}\right) \frac{d^{\ell-2-i}}{du^{\ell-2-i}} H_\ell(u)\\ &= \lim_{u\to q^{-1}} \sum_{i=0}^{\ell-2} \binom{\ell-2}{i} \frac{(-1)^i(\mathfrak{n}+1)\cdots(\mathfrak{n}+i)}{u^{\mathfrak{n}+i+1}} \frac{d^{\ell-2-i}}{du^{\ell-2-i}} H_\ell(u)\\ &= \sum_{i=0}^{\ell-2} \binom{\ell-2}{i} (-1)^i(\mathfrak{n}+1)\cdots(\mathfrak{n}+i) q^{\mathfrak{n}+i+1} \left.\frac{d^{\ell-2-i}}{du^{\ell-2-i}} H_\ell(u)\right|_{u=q^{-1}},\end{aligned}$$

which proves that this residue is given by a polynomial evaluated in $\mathfrak{n}$.

We take a closer look at the main term of this polynomial, which is the dominating term when $\mathfrak{n}\to\infty$. We obtain

$$\begin{aligned}\operatorname{Res}_{u=q^{-1}} \frac{A(u)}{u^{\mathfrak{n}+1}} &= \lim_{u\to q^{-1}} \frac{1}{(\ell-2)!} \frac{(-1)^{\ell-2}(\mathfrak{n}+1)\cdots(\mathfrak{n}+\ell-2)}{u^{\mathfrak{n}+\ell-1}} \left(\frac{(-(1-qu^2)(1+u))^{\ell-1}}{q^{\ell-1}}\right)\\ &\quad\times \prod_{j=1}^{\ell-2}\prod_v \left(1-\frac{ju^{2\deg v}}{(1+u^{\deg v})(1+ju^{\deg v})}\right)(1+O(1/\mathfrak{n}))\\ &= -\frac{\mathfrak{n}^{\ell-2}}{(\ell-2)!}(1-q^{-2})^{\ell-1} q^{\mathfrak{n}} \prod_{j=1}^{\ell-2}\prod_v \left(1-\frac{jq^{-2\deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right)(1+O(1/\mathfrak{n})).\end{aligned}$$

For the other residues, coming from simple poles,

$$\begin{aligned}&\operatorname{Res}_{u=(q\xi_\ell^{j_0})^{-1}} \frac{B(u)}{u^{\mathfrak{n}+1}}\\ &= \lim_{u\to q^{-1}\xi_\ell^{-j_0}} \frac{(u-q^{-1}\xi_\ell^{-j_0})}{u^{\mathfrak{n}+1}} \prod_{j=1}^{\ell-1} \frac{(1-q\xi_\ell^{2j}u^2)(1+\xi_\ell^j u)}{(1-q\xi_\ell^j u)} \prod_v \frac{\left(1+b(v)u^{\deg v}\right)}{\prod_{j=1}^{\ell-1}\left(1+(\xi_\ell^j u)^{\deg v}\right)}\\ &= \lim_{u\to q^{-1}\xi_\ell^{-j_0}} \frac{-(1-q\xi_\ell^{2j_0}u^2)(1+\xi_\ell^{j_0}u)}{u^{\mathfrak{n}+1}q\xi_\ell^{j_0}} \prod_{j=1,j\neq j_0}^{\ell-1} \frac{(1-q\xi_\ell^{2j}u^2)(1+\xi_\ell^j u)}{(1-q\xi_\ell^j u)} \prod_v \frac{\left(1+b(v)u^{\deg v}\right)}{\prod_{j=1}^{\ell-1}\left(1+(\xi_\ell^j u)^{\deg v}\right)}\\ &= -(q\xi_\ell^{j_0})^{\mathfrak{n}}(1-q^{-2}) \prod_{j=1,j\neq j_0}^{\ell-1} \frac{(1-q^{-1}\xi_\ell^{2j-2j_0})(1+q^{-1}\xi_\ell^{j-j_0})}{(1-\xi_\ell^{j-j_0})} \prod_v \frac{\left(1+b(v)(q^{-1}\xi_\ell^{-j_0})^{\deg v}\right)}{\prod_{j=1}^{\ell-1}\left(1+(q^{-1}\xi_\ell^{j-j_0})^{\deg v}\right)}.\end{aligned}$$

We note that the line above is $O(q^{\mathfrak{n}})$ and it contributes to the constant coefficient of $P_\ell(\mathfrak{n})$.

Replacing the residues in (16) with the equations above completes the proof. □

In spite of the fact that Corollary 1.2 can be deduced from the statement of Theorem 1.1, we will prove it first and independently of Theorem 1.1 as a way of introducing the key ideas in the proof of Theorem 1.1.

Proposition 4.2. *Let $v_0 \in \mathcal{V}_K$ be a place of K, let $\epsilon_0 \in \{$ramified, split, inert$\}$, and let $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \epsilon_0)$ be the number of extensions of $\mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the degree of the conductor is equal to $\mathfrak{n}$ and with the prescribed behavior ϵ_0 at the place v_0.*

Then,

$$\begin{aligned} N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \textit{ramified}) &= \frac{(\ell-1)q^{-\deg v_0}}{1+(\ell-1)q^{-\deg v_0}} C_\ell q^{\mathfrak{n}} P_R(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right) \\ N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \textit{split}) &= \frac{1}{\ell(1+(\ell-1)q^{-\deg v_0})} C_\ell q^{\mathfrak{n}} P_S(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right) \\ N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \textit{inert}) &= \frac{1}{\ell(1+(\ell-1)q^{-\deg v_0})} C_\ell q^{\mathfrak{n}} P_I(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right), \end{aligned}$$

where C_ℓ is the non-zero constant defined by (3), $P_R(X)$ and $P_S(X) \in \mathbb{R}[X]$ are monic polynomials of degree $\ell-2$ and $P_I(X) = (\ell-1)P_S(X)$.

Proof. The generating function for the number of extensions counted in $N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n})$ ramified at v_0 is

$$\begin{aligned} \mathcal{F}_R(s) &= \sum_{\substack{\mathrm{Gal}(L/K)\cong\mathbb{Z}/\ell\mathbb{Z} \\ v_0 \text{ ramified}}} \mathfrak{D}(L/K)^{-s} \\ &= (\ell-1)Nv_0^{-(\ell-1)s} \prod_{v\neq v_0} \left(1+(\ell-1)Nv^{-(\ell-1)s}\right) \\ &\quad +(\ell-1)b(v_0)Nv_0^{-(\ell-1)s} \prod_{v\neq v_0} \left(1+b(v)Nv^{-(\ell-1)s}\right) \\ &= \frac{(\ell-1)Nv_0^{-(\ell-1)s}}{1+(\ell-1)Nv_0^{-(\ell-1)s}}\mathcal{A}(s) + (\ell-1)\frac{b(v_0)Nv_0^{-(\ell-1)s}}{1+b(v_0)Nv_0^{-(\ell-1)s}}\mathcal{B}(s). \end{aligned}$$

where we have excluded the case of $\phi_{v_0}(g_{v_0}) = 0$ to account for v_0 ramified.

With the change of variable $u = q^{-(\ell-1)s}$, we obtain

$$F_R(u) = \frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}}A(u) + (\ell-1)\frac{b(v_0)u^{\deg v_0}}{1+b(v_0)u^{\deg v_0}}B(u).$$

Then, applying Theorem 2.1 and Lemma 3.3 with $\delta = \frac{1}{2(\ell-1)} + \varepsilon$ for any $\varepsilon > 0$, we get

$$\begin{aligned} N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \text{ramified}) &= -\mathrm{Res}_{u=q^{-1}} \frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}} \frac{A(u)}{u^{\mathfrak{n}+1}} \\ &\quad -(\ell-1)\sum_{j=1}^{\ell-1} \mathrm{Res}_{u=(q\xi_\ell^j)^{-1}} \frac{b(v_0)u^{\deg v_0}}{1+b(v_0)u^{\deg v_0}} \frac{B(u)}{u^{\mathfrak{n}+1}} \\ &\quad +O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right). \end{aligned}$$

For the residue involving the function $A(u)$, we have

$$\mathrm{Res}_{u=q^{-1}} \frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}} \frac{A(u)}{u^{\mathfrak{n}+1}} = \lim_{u\to q^{-1}} \frac{d^{\ell-2}}{du^{\ell-2}} \frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}} \frac{H_\ell(u)}{u^{\mathfrak{n}+1}},$$

where $H_\ell(u)$ is given by (17). This yields

$$\begin{aligned}
&\mathrm{Res}_{u=q^{-1}}\frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}}\frac{A(u)}{u^{\mathfrak{n}+1}}\\
&=\sum_{i=0}^{\ell-2}\binom{\ell-2}{i}(-1)^i(\mathfrak{n}+1)\cdots(\mathfrak{n}+i)q^{\mathfrak{n}+i+1}\ \frac{d^{\ell-2-i}}{du^{\ell-2-i}}\frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}}H_\ell(u)\Big|_{u=q^{-1}},
\end{aligned}$$

and we obtain the polynomial in $\mathfrak{n}$ as in the case of the proof of Theorem 4.1. As before, we record the main coefficient as the term dominating when $\mathfrak{n}\to\infty$ to be

$$\begin{aligned}
&\mathrm{Res}_{u=q^{-1}}\frac{(\ell-1)u^{\deg v_0}}{1+(\ell-1)u^{\deg v_0}}\frac{A(u)}{u^{\mathfrak{n}+1}}\\
&=-\frac{\mathfrak{n}^{\ell-2}}{(\ell-2)!}(1-q^{-2})^{\ell-1}q^{\mathfrak{n}}\frac{(\ell-1)q^{-\deg v_0}}{1+(\ell-1)q^{-\deg v_0}}\prod_{j=1}^{\ell-2}\prod_v\left(1-\frac{jq^{-2\deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right)(1+O(1/\mathfrak{n}))\\
&=-C_\ell\frac{(\ell-1)q^{-\deg v_0}}{1+(\ell-1)q^{-\deg v_q}}q^{\mathfrak{n}}\mathfrak{n}^{\ell-2}(1+O(1/\mathfrak{n})).
\end{aligned}$$

For the residues involving the function $B(u)$, we notice that, since the poles are of order one,

$$\mathrm{Res}_{u=(q\xi_\ell^j)^{-1}}\frac{b(v_0)u^{\deg v_0}}{1+b(v_0)u^{\deg v_0}}\frac{B(u)}{u^{\mathfrak{n}+1}}=\frac{b(v_0)(q\xi_\ell^j)^{-\deg v_0}}{1+b(v_0)(q\xi_\ell^j)^{-\deg v_0}}\mathrm{Res}_{u=(q\xi_\ell^j)^{-1}}\frac{B(u)}{u^{\mathfrak{n}+1}}.$$

The number above is equal to $O(q^{\mathfrak{n}})$ and it will contribute to the constant coefficient of the polynomial $P_R(\mathfrak{n})$. This proves the result for the number of extensions ramifying at v_0. We now consider the case of extensions splitting at v_0. First, we write the generating function for the number of extensions of K unramified at v_0 is, up to a constant,

$$\begin{aligned}
\mathcal{F}_U(s)&=\ell+\sum_{\substack{\mathrm{Gal}(L/K)\cong\mathbb{Z}/\ell\mathbb{Z}\\ v_0\text{ unramified}}}\mathfrak{D}(L/K)^{-s}\\
&=\sum_{j=0}^{\ell-1}\prod_{v\neq v_0}\left(1+\left(\xi_\ell^{j\deg v}+\cdots+\xi_\ell^{(\ell-1)j\deg v}\right)Nv^{-(\ell-1)s}\right)\\
&=\prod_{v\neq v_0}\left(1+(\ell-1)Nv^{-(\ell-1)s}\right)+(\ell-1)\prod_{v\neq v_0}\left(1+b(v)Nv^{-(\ell-1)s}\right)\\
&=\frac{1}{1+(\ell-1)Nv_0^{-(\ell-1)s}}\mathcal{A}(s)+\frac{(\ell-1)}{1+b(v_0)Nv_0^{-(\ell-1)s}}\mathcal{B}(s).
\end{aligned}$$

Using the notation of Section 3, recall that $b_\ell=\mu^{\frac{q-1}{\ell}}$ where μ is a generator of $\mathbb{F}_q^\times$ (hence b_ℓ is an ℓth root of unity in $\mathbb{F}_q^\times$), and $\sigma:\mathbb{F}_q^\times\to\mathbb{C}$ is a character of order ℓ. Let $\rho_\ell=\sigma(b_\ell)$, which is then a primitive ℓth root of unity in $\mathbb{C}$. For each v, denote by n_v the positive integer such that the image of v_0 in $(\mathcal{O}_v/(\pi_v))^\times$ is $g_v^{n_v}$. Then $\phi_v(v_0)=n_v\phi_v(g_v)$. Hence by (8) v_0 is unramified and split if and only if $\phi_{v_0}(\mathcal{O}_{v_0}^\times)=0$ and

$$-(\deg v_0)\psi_\infty(\pi_\infty)+\sum_{v\neq v_0,v_\infty}n_v\phi_v(g_v)\equiv 0\pmod{\ell}$$

which is equivalent to

$$\rho_\ell^{-\deg v_0 \psi_\infty(\pi_\infty)} \prod_{v\neq v_0,v_\infty} \rho_\ell^{n_v\phi_v(g_v)} = 1$$

for the primitive ℓth root of unity ρ_ℓ coming from the choice of primitive root $b_\ell \in \mathbb{F}_q^\times$ that we fixed in Section 3.

Thus $v_0 \neq v_\infty$ is unramified and split if and only if $\phi_{v_0}(\mathcal{O}_{v_0}^\times) = 0$ and

$$D(v_0) := \rho_\ell^{-\deg v_0 \psi_\infty(\pi_\infty)} \prod_{v\notin\{v_0,v_\infty\}} \chi_{v,\ell}(v_0)^{\phi_v(g_v)} = 1. \tag{18}$$

Since $D(v_0)$ is a ℓth root of unity, we can rewrite (18) as

$$\frac{1}{\ell}\sum_{j=0}^{\ell-1} D(v_0)^j = \begin{cases} 1 & \text{if } v_0 \text{ is unramified and split,} \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

and this is the criterion that we will use in the generating series.

Analogously, we also have that v_∞ is unramified and split if and only if $\phi_{v_\infty}(\mathcal{O}_{v_\infty}^\times) = 0$ and

$$\rho_\ell^{-\deg v_\infty \psi_\infty(\pi_\infty)} = 1,$$

since $\deg v_\infty = 1$.

We claim that the Dirichlet series for cyclic extensions splitting at a fixed place $v_0 \neq v_\infty$ is

$$\begin{aligned} \mathcal{F}_S(s) \;\; = \;\; & \frac{1}{\ell^2}\sum_{j=0}^{\ell-1}\sum_{k=0}^{\ell-1}\sum_{r=0}^{\ell-1} \rho_\ell^{-rk\deg v_0} \\ & \times \prod_{v\neq v_0,v_\infty} \left(1 + (\xi_\ell^{j\deg v}\chi_{v,\ell}(v_0)^k + \cdots + \xi_\ell^{(\ell-1)j\deg v}\chi_{v,\ell}(v_0)^{(\ell-1)k})Nv^{-(\ell-1)s}\right) \\ & \times \left(1 + (\xi_\ell^{j\deg v_\infty} + \cdots + \xi_\ell^{(\ell-1)j\deg v_\infty})Nv_\infty^{-(\ell-1)s}\right). \end{aligned}$$

Recall from Section 2 that we have to count the maps $\phi : \pi_\infty^{\mathbb{Z}} \times \prod_v \mathcal{O}_v^\times \to \mathbb{Z}/\ell\mathbb{Z}$ satisfying (7), together with the splitting conditions (18) and $\phi_{v_0}(\mathcal{O}_{v_0}^\times) = 0$. Let $\mathrm{Cond}(\phi)$ be the conductor of such a map ϕ, and v a place of the conductor. For each fixed j, k, r in the first line above, the ith term $\rho_\ell^{-rk\deg v_0}\xi_\ell^{ij\deg v}\chi_{v,\ell}(v_0)^{ik}Nv^{-(\ell-1)s}$ in the Euler product corresponds to the map where $\phi_v(g_v) = i$ and $\psi_\infty(\pi_\infty) = r$, for $1 \leq i \leq \ell-1$. Considering all the places v of $\mathrm{Cond}(\phi)$ (including v_∞, which is accounted for in the second line of the equation), the term in the j,k,rth Dirichlet series above corresponding to the global map ϕ equals

$$\left(\xi_\ell^{\sum_v j\phi_v(g_v)\deg v}\right) \times \rho_\ell^{-rk\deg v_0} \prod_{v\neq v_0,v_\infty} \chi_{v,\ell}(v_0)^{k\phi_v(g_v)} \times N(\mathrm{Cond}(\phi))^{-(\ell-1)s}.$$

Summing over j, we obtain zero unless condition (7) is satisfied. Summing over r covers all the possible values of $\psi_\infty(\pi_\infty)$. Finally, summing over k yields zero unless condition (18) is satisfied. Thus the sum of those terms over r, k, j, together with the correcting factor $\frac{1}{\ell^2}$ will yield $\ell N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$ if both conditions (7) and (18) are satisfied and zero otherwise.

We also remark that the constant term of $\mathcal{F}_S(s)$ is ℓ if $\ell \mid \deg v_0$ and 1 otherwise.

When $v_0 = v_\infty$, we have,

$$\begin{aligned}\mathcal{F}_S(s) &= \frac{1}{\ell^2}\sum_{j=0}^{\ell-1}\sum_{k=0}^{\ell-1}\sum_{r=0}^{\ell-1}\rho_\ell^{-rk\deg v_\infty}\prod_{v\neq v_\infty}\left(1+(\xi_\ell^{j\deg v}+\cdots+\xi_\ell^{(\ell-1)j\deg v})Nv^{-(\ell-1)s}\right)\\ &= \frac{1}{\ell}\sum_{j=0}^{\ell-1}\prod_{v\neq v_\infty}\left(1+(\xi_\ell^{j\deg v}+\cdots+\xi_\ell^{(\ell-1)j\deg v})Nv^{-(\ell-1)s}\right) = \frac{1}{\ell}\mathcal{F}_U(s).\end{aligned}$$

We note that since v_0 is split in the first case, and v_∞ is split in the second case, $\chi_{v_\infty}(v_0) = 1$ in the first case and $\chi_{v_0}(v_\infty) = 1$ in the second case. This results in the following symmetric formula

$$\begin{aligned}\mathcal{F}_S(s) &= \frac{1}{\ell^2}\sum_{j=0}^{\ell-1}\sum_{k=0}^{\ell-1}\sum_{r=0}^{\ell-1}\rho_\ell^{-rk\deg v_0}\\ &\quad\times\prod_{v\neq v_0}\left(1+(\xi_\ell^{j\deg v}\chi_{v,\ell}(v_0)^k+\cdots+\xi_\ell^{(\ell-1)j\deg v}\chi_{v,\ell}(v_0)^{(\ell-1)k})Nv^{-(\ell-1)s}\right)\end{aligned}$$

which is valid for any place v_0.

Separating the term with $k = 0$ from the terms with $k \neq 0$, we obtain,

$$\begin{aligned}\mathcal{F}_S(s) &= \frac{1}{\ell}\mathcal{F}_U(s)\\ &\quad+\frac{1}{\ell^2}\sum_{j=0}^{\ell-1}\sum_{k=1}^{\ell-1}\left(\sum_{r=0}^{\ell-1}\rho_\ell^{-rk\deg v_0}\right)\mathcal{M}_{j,k}(s, v_0, \text{split}),\end{aligned}$$

where $\mathcal{M}_{j,k}(s, v_0, \text{split})$ is given by (12).

Applying Theorem 2.1, and Lemmas 3.2 and 3.3 to the generating function $\mathcal{F}_S(s)$, we get

$$\begin{aligned}N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \text{split}) &= -\frac{1}{\ell}\text{Res}_{u=q^{-1}}\frac{1}{1+(\ell-1)u^{\deg v_0}}\frac{A(u)}{u^{\mathfrak{n}+1}}\\ &\quad-\frac{\ell-1}{\ell}\sum_{j=1}^{\ell-1}\text{Res}_{u=(\xi_\ell^j q)^{-1}}\frac{1}{\ell(1+b(v_0)u^{\deg v_0})}\frac{B(u)}{u^{\mathfrak{n}+1}}\\ &\quad+O\left(q^{(1/2+\varepsilon)\mathfrak{n}}\right).\end{aligned}$$

As before, the residue involving the function $A(u)$ yields $q^{\mathfrak{n}}$ times a polynomial in $\mathfrak{n}$ of degree $\ell - 2$. The main term when $\mathfrak{n}$ goes infinity is given by the leading term of the polynomial, and is

$$\text{Res}_{u=q^{-1}}\frac{1}{\ell(1+(\ell-1)u^{\deg v_0})}\frac{A(u)}{u^{\mathfrak{n}+1}} = -C_\ell q^{\mathfrak{n}}\mathfrak{n}^{\ell-2}\frac{1}{\ell(1+(\ell-1)q^{-\deg v_0})}.$$

Similarly the value of

$$\text{Res}_{u=(\xi_\ell^j q)^{-1}}\frac{1}{\ell(1+b(v_0)u^{\deg v_0})}\frac{B(u)}{u^{\mathfrak{n}+1}}$$

is $O(q^{\mathfrak{n}})$ and it contributes to the constant coefficient of $P_S(\mathfrak{n})$.

We now consider the Dirichlet series for cyclic extensions for which a fixed place v_0 is inert. It is given by

$$\begin{aligned}
\mathcal{F}_I(s) &= \mathcal{F}_U(s) - \mathcal{F}_I(s) \\
&= \sum_{j=0}^{\ell-1} \prod_{v \neq v_0} \left(1 + \left(\xi_\ell^{j \deg v} + \cdots + \xi_\ell^{(\ell-1)j \deg v}\right) Nv^{-(\ell-1)s}\right) \\
&- \frac{1}{\ell^2} \sum_{j=0}^{\ell-1} \sum_{k=0}^{\ell-1} \sum_{r=0}^{\ell-1} \rho_\ell^{-rk \deg v_0} \prod_{v \neq v_0} \left(1 + (\xi_\ell^{j \deg v} \chi_{v,\ell}(v_0)^k + \cdots + \xi_\ell^{(\ell-1)j \deg v} \chi_{v,\ell}(v_0)^{(\ell-1)k}) Nv^{-(\ell-1)s}\right) \\
&= \frac{(\ell-1)}{\ell} \sum_{j=0}^{\ell-1} \prod_{v \neq v_0} \left(1 + \left(\xi_\ell^{j \deg v} + \cdots + \xi_\ell^{(\ell-1)j \deg v}\right) Nv^{-(\ell-1)s}\right) \\
&- \frac{1}{\ell^2} \sum_{j=0}^{\ell-1} \sum_{k=1}^{\ell-1} \sum_{r=0}^{\ell-1} \rho_\ell^{-rk \deg v_0} \prod_{v \neq v_0} \left(1 + (\xi_\ell^{j \deg v} \chi_{v,\ell}(v_0)^k + \cdots + \xi_\ell^{(\ell-1)j \deg v} \chi_{v,\ell}(v_0)^{(\ell-1)k}) Nv^{-(\ell-1)s}\right).
\end{aligned}$$

The main term is given by

$$\frac{(\ell-1)}{\ell} \mathcal{F}_U(s) = \frac{(\ell-1)}{\ell} \left(\frac{1}{1 + (\ell-1)\, N v_0^{-(\ell-1)s}} \mathcal{A}(s) + \frac{\ell - 1}{1 + b(v_0) N v_0^{-(\ell-1)s}} \mathcal{B}(s) \right).$$

The proof proceeds exactly as in the split case. In particular, this proves that

$$N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \text{inert}) = (\ell-1) N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}, v_0, \text{split}) + O\left(q^{(1/2+\varepsilon)\mathfrak{n}}\right).$$

This concludes the proof of the Proposition 4.2. $\square$

We are now ready to prove a more general statement.

Theorem 4.3. *Let $\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I$ be three finite and disjoint sets of places of K. Let*

$$N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)$$

be the number of extensions of $\mathbb{F}_q(X)$ with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ such that the degree of the conductor is $\mathfrak{n}$, and which are ramified at the places of $\mathcal{V}_R$, (completely) split at the places of $\mathcal{V}_S$ and inert at the places of $\mathcal{V}_I$. Let $\mathcal{V} = \mathcal{V}_R \cup \mathcal{V}_S \cup \mathcal{V}_I$. Then,

$$\begin{aligned}
N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I) &= C_\ell \left(\prod_{v \in \mathcal{V}} c_v \right) q^{\mathfrak{n}} P_{\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I}(\mathfrak{n}) + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)\mathfrak{n}}\right) \\
\frac{N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n}; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)}{N(\mathbb{Z}/\ell\mathbb{Z}, \mathfrak{n})} &= \left(\prod_{v \in \mathcal{V}} c_v \right) \left(1 + O\left(\frac{1}{\mathfrak{n}}\right)\right),
\end{aligned}$$

where $P_{\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I}(X) \in \mathbb{R}[X]$ is a monic polynomial of degree $\ell - 2$ and C_ℓ is given by

$$C_\ell = \frac{(1-q^{-2})^{\ell-1}}{(\ell-2)!} \prod_{j=1}^{\ell-2} \prod_{v \in \mathcal{V}_K} \left(1 - \frac{j q^{-2 \deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right).$$

In addition,

$$c_{v_0} = \begin{cases} \dfrac{(\ell-1)q^{-\deg v_0}}{1+(\ell-1)q^{-\deg v_0}} & \text{if } v_0 \in \mathcal{V}_R, \\ \dfrac{1}{\ell(1+(\ell-1)q^{-\deg v_0})} & \text{if } v_0 \in \mathcal{V}_S, \\ \dfrac{\ell-1}{\ell(1+(\ell-1)q^{-\deg v_0})} & \text{if } v_0 \in \mathcal{V}_I. \end{cases}$$

Proof. Let $\mathcal{V}_U = \mathcal{V}_S \cup \mathcal{V}_I$.

We first construct the Dirichlet generating series with prescibed conditions for $\mathcal{V}_R$, $\mathcal{V}_U$, and $\mathcal{V}_S = \{v_1, \dots, v_n\} \subset \mathcal{V}_U$. In other words, for the elements $v \in \mathcal{V}_I$ we will only prescribe that they are in $\mathcal{V}_U$ and we will ignore the inert condition for the moment. We claim that the generating series is then

$$\begin{aligned} &\mathcal{F}_{\mathcal{V}_R, \mathcal{V}_S \subset \mathcal{V}_U}(s) \\ &= \frac{1}{\ell^{n+1}} \sum_{j=0}^{\ell-1} \sum_{k_1=0}^{\ell-1} \cdots \sum_{k_n=0}^{\ell-1} \sum_{r=0}^{\ell-1} \rho_\ell^{-r\sum_{h=1}^n k_h \deg v_h} \\ &\times \prod_{v \notin \mathcal{V}_R \cup \mathcal{V}_U} \left(1 + \left(\xi_\ell^{j\deg v} \prod_{h=1}^n \chi_{v,\ell}(v_h)^{k_h} + \cdots + \xi_\ell^{(\ell-1)j\deg v} \prod_{h=1}^n \chi_{v,\ell}(v_h)^{(\ell-1)k_h}\right) Nv^{-(\ell-1)s}\right) \\ &\times \prod_{v \in \mathcal{V}_R} \left(\xi_\ell^{j\deg v} + \cdots + \xi_\ell^{(\ell-1)j\deg v}\right) Nv^{-(\ell-1)s}. \end{aligned}$$

Let us check that the formula above is correct. We want to count the maps ϕ with parameters $(\{r_v\}, r)$ which are ramified at the primes of $\mathcal{V}_R$, unramified at the primes of $\mathcal{V}_U$ and split at the primes of $\mathcal{V}_S$. Assume that we have $0 < r_v \leq \ell - 1$ for all $v \in \mathcal{V}_R$, and $r_v = 0$ for all primes of $\mathcal{V}_U$. For each fixed $j, k_1, \dots, k_n$, the map ϕ with parameters $(\{r_v\}, r)$ corresponds to the component

$$\left(\prod_{v \notin \mathcal{V}_R \cup \mathcal{V}_U} \rho_\ell^{-r\sum_{h=1}^n k_h \deg v_h} \xi_\ell^{jr_v \deg v} \prod_{h=1}^n \chi_{v,\ell}(v_h)^{r_v k_h}\right) \times \left(\prod_{v \in \mathcal{V}_R} \xi_\ell^{r_v j \deg v}\right) \times N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$$

of the Euler product. Summing over all $j, k_1, \dots, k_n$, we get that the coefficient of $N(\mathrm{Cond}(\phi))^{-(\ell-1)s}$ is given by

$$\begin{aligned} &\left(\sum_{k_1=0}^{\ell-1} \rho_\ell^{-rk_1 \deg v_1} \prod_v \chi_{v,\ell}(v_1)^{r_v k_1}\right) \times \cdots \times \left(\sum_{k_n=0}^{\ell-1} \rho_\ell^{-rk_n \deg v_n} \prod_v \chi_{v,\ell}(v_n)^{r_v k_n}\right) \\ &\times \left(\sum_{j=0}^{\ell-1} \xi_\ell^{j\sum_{v|\mathrm{Cond}(\phi)} r_v \deg v}\right) \\ &= \begin{cases} \ell^{n+1} & \text{if } \sum_{v|\mathrm{Cond}(\phi)} r_v \deg v \equiv 0 \pmod{\ell} \text{ and } \phi \text{ is split at } v_1, \dots, v_n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We now write the generating series as $\mathcal{F}_{\mathcal{V}_R, \mathcal{V}_S \subset \mathcal{V}_U}(s) = \mathcal{F}^1_{\mathcal{V}_R, \mathcal{V}_S \subset \mathcal{V}_U}(s) + \mathcal{F}^2_{\mathcal{V}_R, \mathcal{V}_S \subset \mathcal{V}_U}(s)$, where the first series contributes to the main term and the second to the error term. Taking

$(k_1,\ldots,k_n)=(0,\ldots,0)$ in $\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S\subset\mathcal{V}_U}(s)$, we have

$$\begin{aligned}\mathcal{F}^1_{\mathcal{V}_R,\mathcal{V}_S\subset\mathcal{V}_U}(s) &= \frac{1}{\ell^n}\Bigg(\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)Nv^{-(\ell-1)s}}{1+(\ell-1)Nv^{-(\ell-1)s}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)Nv^{-(\ell-1)s}}\mathcal{A}(s)\\ &\quad +(\ell-1)\prod_{v\in\mathcal{V}_R}\frac{b(v)Nv^{-(\ell-1)s}}{1+b(v)Nv^{-(\ell-1)s}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+b(v)Nv^{-(\ell-1)s}}\mathcal{B}(s)\Bigg),\end{aligned}$$

where as usual $j=0$ gives the function $\mathcal{A}(s)$ defined by (13) and the other values of j give $\ell-1$ copies of the function $\mathcal{B}(s)$ defined by (14).

Taking $(k_1,\ldots,k_n)\neq(0,\ldots,0)$ in $\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S\subset\mathcal{V}_U}(s)$, we have

$$\mathcal{F}^2_{\mathcal{V}_R,\mathcal{V}_S\subset\mathcal{V}_U}(s) = \frac{1}{\ell^{n+1}}\sum_{j=0}^{\ell-1}\sum_{\substack{k_1,\ldots,k_n=0\\(k_1,\ldots,k_n)\neq(0,\ldots,0)}}^{\ell-1} G(s)\ M_{j,k_1,\ldots,k_n}(s;\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_U)$$

where

$$G(s)=\sum_{r=0}^{\ell-1}\rho_\ell^{-r\sum_{h=1}^n k_h\deg v_h}\prod_{v\in\mathcal{V}_R}\left(\xi_\ell^{j\deg v}+\cdots+\xi_\ell^{(\ell-1)j\deg v}\right)Nv^{-(\ell-1)s}$$

is analytic for all $s\in\mathbb{C}$, and where for each fixed vector $(k_1,\ldots,k_n)\neq(0,\ldots,0)$, and for each $0\le j\le\ell-1$, we have that

$$\begin{aligned}&\mathcal{M}_{j,k_1,\ldots,k_n}(s;\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_U)\\ &= \prod_{v\notin\mathcal{V}_R\cup\mathcal{V}_U}\left(1+\left(\xi_\ell^{j\deg v}\prod_{h=1}^n\chi_{v,\ell}(v_h)^{k_h}+\cdots+\xi_\ell^{(\ell-1)j\deg v}\prod_{h=1}^n\chi_{v,\ell}(v_h)^{(\ell-1)k_h}\right)Nv^{-(\ell-1)s}\right).\end{aligned}$$

Let $N'(\mathbb{Z}/\ell\mathbb{Z},\mathfrak{n};\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)$ be the number of extensions where the degree of the conductor is $\mathfrak{n}$ and with the prescribed ramification conditions at the primes of $\mathcal{V}_R$ and $\mathcal{V}_S$, and unramified at the primes of $\mathcal{V}_I$, i.e. the extensions counted by the generating series $\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S\subset\mathcal{V}_U}(s)$ above. By Theorem 2.1, and Lemmas 3.2 and 3.3,

$$\begin{aligned}&N'(\mathbb{Z}/\ell\mathbb{Z},\mathfrak{n};\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)=\\ &-\frac{1}{\ell^n}\Bigg(\mathrm{Res}_{u=q^{-1}}\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)u^{\deg v}}{1+(\ell-1)u^{\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)u^{\deg v}}\frac{A(u)}{u^{\mathfrak{n}+1}}\\ &+(\ell-1)\sum_{j=1}^{\ell-1}\mathrm{Res}_{u=(\xi_\ell^j q)^{-1}}\prod_{v\in\mathcal{V}_R}\frac{b(v)u^{\deg v}}{1+b(v)u^{\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+b(v)u^{\deg v}}\frac{B(u)}{u^{\mathfrak{n}+1}}\Bigg)\\ &+O\left(q^{(1/2+\varepsilon)\mathfrak{n}}\right).\end{aligned}$$

As before, the residue involving the function $A(u)$ yields $q^{\mathfrak{n}}$ times a polynomial in $\mathfrak{n}$ of degree $\ell-2$, and the residues of $B(u)$ are $O(q^{\mathfrak{n}})$, so they contribute to the constant coefficient of the polynomial, and not to the main term. The main term when $\mathfrak{n}$ tends to infinity is then given by the leading term of the polynomial which is

$$-\frac{1}{\ell^n}\left(\mathrm{Res}_{u=q^{-1}}\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)u^{\deg v}}{1+(\ell-1)u^{\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)u^{\deg v}}\frac{A(u)}{u^{\mathfrak{n}+1}}\right)$$

$$=\frac{1}{\ell^n}\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)q^{-\deg v}}{1+(\ell-1)q^{-\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)q^{-\deg v}}\;C_\ell q^{\mathfrak{n}}\mathfrak{n}^{\ell-2}. \tag{20}$$

We now proceed to add the conditions at the primes of $\mathcal{V}_I=\mathcal{V}_U\setminus\mathcal{V}_S$. Using inclusion-exclusion, it is easy to see that

$$N(\mathbb{Z}/\ell\mathbb{Z},\mathfrak{n};\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)=\sum_{\tilde{\mathcal{V}}_I\subset\mathcal{V}_I}(-1)^{|\tilde{\mathcal{V}}_I|}N'(\mathbb{Z}/\ell\mathbb{Z},\mathfrak{n};\mathcal{V}_R,\mathcal{V}_S\cup\tilde{\mathcal{V}}_I,\mathcal{V}_I\setminus\tilde{\mathcal{V}}_I). \tag{21}$$

We can rewrite this in terms of generating series. Let $\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I}(s)$ be the generating series for the extensions counted by $N(\mathbb{Z}/\ell\mathbb{Z},\mathfrak{n};\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I)$. Then, it follows from (21) that

$$\begin{aligned}\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S,\mathcal{V}_I}(s) &= \sum_{\tilde{\mathcal{V}}_I\subset\mathcal{V}_I}(-1)^{|\tilde{\mathcal{V}}_I|}\mathcal{F}_{\mathcal{V}_R,\mathcal{V}_S\cup\tilde{\mathcal{V}}_I\subset\mathcal{V}_U}(s)\\ &= \sum_{\tilde{\mathcal{V}}_I\subset\mathcal{V}_I}(-1)^{|\tilde{\mathcal{V}}_I|}\left(\mathcal{F}^1_{\mathcal{V}_R,\mathcal{V}_S\cup\tilde{\mathcal{V}}_I\subset\mathcal{V}_U}(s)+\mathcal{F}^2_{\mathcal{V}_R,\mathcal{V}_S\cup\tilde{\mathcal{V}}_I\subset\mathcal{V}_U}(s)\right),\end{aligned}$$

and the main term will be given by the sum of the poles of the generating series $\mathcal{F}^1_{\mathcal{V}_R,\mathcal{V}_S\cup\tilde{\mathcal{V}}_I\subset\mathcal{V}_U}(s)$. Using (20), this is given by

$$\begin{aligned}&C_\ell q^{\mathfrak{n}}\mathfrak{n}^{\ell-2}\left(\sum_{\tilde{\mathcal{V}}_I\subset\mathcal{V}_I}\frac{(-1)^{|\tilde{\mathcal{V}}_I|}}{\ell^{|\mathcal{V}_S|\cup|\tilde{\mathcal{V}}_I|}}\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)q^{-\deg v}}{1+(\ell-1)q^{-\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)q^{-\deg v}}\right)\\ =\;& C_\ell q^{\mathfrak{n}}\mathfrak{n}^{\ell-2}\left(\frac{1}{\ell}\right)^{|\mathcal{V}_S|}\left(\frac{\ell-1}{\ell}\right)^{|\mathcal{V}_I|}\prod_{v\in\mathcal{V}_R}\frac{(\ell-1)q^{-\deg v}}{1+(\ell-1)q^{-\deg v}}\prod_{v\in\mathcal{V}_U}\frac{1}{1+(\ell-1)q^{-\deg v}}\\ =\;& C_\ell\left(\prod_{v\in\mathcal{V}_R\cup\mathcal{V}_S\cup\mathcal{V}_I}c_v\right)q^{\mathfrak{n}}\mathfrak{n}^{\ell-2},\end{aligned}$$

where the c_v are as in Theorem 4.3.

Dividing the last line by (15) completes the proof of the statement. □

4.1. **Quadratic extensions.** We now look specifically at the case $\ell=2$ as we obtain the number of quadratic extensions of K with conductor $\mathfrak{n}$ with no error term, and the ramified case with a better error term without using the Tauberian theorem. The generating function $\mathcal{F}$ becomes

$$\mathcal{F}(s)=2+\sum_{\mathrm{Gal}(L/K)\cong\mathbb{Z}/2\mathbb{Z}}\mathfrak{D}(L/K)^{-s}=\prod_v\left(1+Nv^{-s}\right)+\prod_v\left(1+(-1)^{\deg v}Nv^{-s}\right).$$

In this case,

$$\begin{aligned}\mathcal{A}(s) = \prod_v \left(1 + Nv^{-s}\right) &= \prod_v \frac{(1 - Nv^{-2s})}{(1 - Nv^{-s})} \\ &= \frac{\zeta_K(s)}{\zeta_K(2s)} = \frac{(1 - q^{1-2s})(1 + q^{-s})}{1 - q^{1-s}}.\end{aligned}$$

After making the change of variables $u = q^{-s}$, we obtain

$$A(u) := \frac{(1 - qu^2)(1 + u)}{1 - qu}.$$

Analogously,

$$\mathcal{B}(s) = \prod_v \left(1 + (-1)^{\deg v} Nv^{-s}\right) = \frac{(1 - q^{1-2s})(1 - q^{-s})}{1 + q^{1-s}}$$

which equals $A(-u)$ after the change of variables $u = q^{-s}$. Then,

$$\begin{aligned} F(u) &= A(u) + A(-u) \\ &= (1 - qu^2)\left(\frac{1+u}{1-qu} + \frac{1-u}{1+qu}\right). \end{aligned}$$

By identifying the coefficients in the power series expansion in u of the above rational function for $\mathfrak{n} > 0$ with the coefficients of

$$2 + \sum_{\mathfrak{n}=1}^{\infty} N(\mathbb{Z}/2\mathbb{Z}, \mathfrak{n}) u^{\mathfrak{n}},$$

we finally obtain that

$$\begin{aligned} N(\mathbb{Z}/2\mathbb{Z}, \mathfrak{n}) &= \begin{cases} \left(1 + (-1)^{\mathfrak{n}}\right)\left(q^{\mathfrak{n}} - q^{\mathfrak{n}-2}\right) & \mathfrak{n} \geq 3 \\ 2q^2 & \mathfrak{n} = 2 \\ 0 & \mathfrak{n} = 1 \end{cases} \\ &= \begin{cases} 2(q^{\mathfrak{n}} - q^{\mathfrak{n}-2}) & \mathfrak{n} > 2, \mathfrak{n} \text{ even}, \\ 2q^2 & \mathfrak{n} = 2, \\ 0 & \mathfrak{n} \text{ odd}. \end{cases} \end{aligned} \tag{22}$$

Remark 4.4. Recall that the number of square-free monic polynomials of degree $d > 1$ is $q^d - q^{d-1}$. In this case, we are counting twice the number of square-free monic polynomials. The counting happens twice since every monic square-free polynomial f gives two quadratic extensions corresponding to $K(\sqrt{f}\,)$ and $K(\sqrt{\beta f}\,)$ where β is a non-square in $\mathbb{F}_q^\times$.

We now proceed to the ramified case.

$$\begin{aligned} F_R(u) &= \frac{u^{\deg v_0}}{1 + u^{\deg v_0}} A(u) + \frac{(-u)^{\deg v_0}}{1 + (-u)^{\deg v_0}} A(-u) \\ &= (1 - qu^2)\left(\frac{u^{\deg v_0}(1+u)}{(1 + u^{\deg v_0})(1 - qu)} + \frac{(-u)^{\deg v_0}(1-u)}{(1 + (-u)^{\deg v_0})(1 + qu)}\right). \end{aligned}$$

We have

$$\begin{aligned} A(u) &= (1-qu^2)\frac{1+u}{1-qu} \\ &= 1+(q+1)u+q^2u^2+\sum_{n=3}^{\infty}(q^n-q^{n-2})u^n. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{u^{\deg v_0}}{1+u^{\deg v_0}}A(u) &= \left(\sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0}\right)\left(1+(q+1)u+q^2u^2+\sum_{n=3}^{\infty}(q^n-q^{n-2})u^n\right) \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0}+(q+1)\sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0+1}+q^2\sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0+2} \\ &\quad+\sum_{k=1}^{\infty}\sum_{n=3}^{\infty}(-1)^{k-1}(q^n-q^{n-2})u^{k\deg v_0+n} \\ &= \sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0}+(q+1)\sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0+1}+q^2\sum_{k=1}^{\infty}(-1)^{k-1}u^{k\deg v_0+2} \\ &\quad+\sum_{m=3+\deg v_0}^{\infty}\sum_{k=1}^{\lfloor\frac{m-3}{\deg v_0}\rfloor}(-1)^{k-1}(q^{m-k\deg v_0}-q^{m-k\deg v_0-2})u^m \\ &= \sum_{m=3+\deg v_0}^{\infty}\frac{1-q^{-2}}{1+q^{-\deg v_0}}q^{m-\deg v_0}u^m+O_q(1)\sum_{m=\deg v_0}^{\infty}u^m. \end{aligned}$$

By identifying the coefficients of $F_R(u)$ with the power series

$$\sum_{\mathfrak{n}=1}^{\infty}N(\mathbb{Z}/2\mathbb{Z},\mathfrak{n},v_0,\text{ramified})u^{\mathfrak{n}},$$

we obtain,

$$N(\mathbb{Z}/2\mathbb{Z},\mathfrak{n},v_0,\text{ramified})=\frac{(1-q^{-2})}{1+q^{-\deg v_0}}q^{\mathfrak{n}-\deg v_0}+O_q(1).$$

5. Distribution of the number of points on covers

We explain in this section how the results of this paper apply to the distribution for the number of $\mathbb{F}_q$-points on covers C on the moduli space $\mathcal{H}_{g,\ell}$.

Consider an ℓ-cyclic cover $C\to\mathbb{P}^1$ defined over $\mathbb{F}_q$ and let L be the function field of C. As mentioned in Section 2.2 the genus g_C of the cover C is related to the discriminant $\mathrm{Disc}(L/K)$ via

$$2g_C=(\ell-1)\left[-2+2\,\mathrm{Disc}(L/K)\right],$$

which implies

$$\mathfrak{n}=\frac{2g_C}{\ell-1}+2, \tag{23}$$

where $\mathfrak{n}$ is the degree of $\mathrm{Cond}(L/K)$.

Recall that the zeta function of a curve C is given by

$$Z_C(u) = \exp\left(\sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n})\frac{u^n}{n}\right). \tag{24}$$

Moreover,

$$Z_L(u) = Z_C(u)$$

with the usual identification $u = q^{-s}$.

We recall that $\mathcal{V}_K$ is the set of places of K. Suppose that L/K is a Galois extension. We can write

$$Z_L(u) = \prod_{v\in\mathcal{V}_K} \left(1 - u^{f(v)\deg v}\right)^{-r(v)}, \tag{25}$$

where for each prime v, we denote by $e(v), f(v), r(v)$ the ramification degree, the inertia degree and the number of primes of L above v respectively.

Taking logarithm on both sides of the equality $Z_C(u) = Z_L(u)$ using (24) and (25), we get

$$\sum_{n=0}^{\infty} \#C(\mathbb{F}_{q^n})\frac{u^n}{n} = \sum_{v\in\mathcal{V}_K}\sum_{m=1}^{\infty} r(v)\frac{u^{mf(v)\deg v}}{m}.$$

Equating the coefficients of u^n on both sides gives

$$\#C(\mathbb{F}_{q^n}) = \sum_{\substack{v\in\mathcal{V}_K\\ f(v)\deg v|n}} r(v)f(v)\deg v. \tag{26}$$

Note that formula (25) implies that the fiber above an $\mathbb{F}_q$-point of $\mathbb{P}^1$ that corresponds to the place v of degree 1 of K contains

$$\begin{cases} \ell \text{ distinct } \mathbb{F}_q\text{-points,} & \text{if } v \text{ splits completely,}\\ 1\ \mathbb{F}_q\text{-point,} & \text{if } v \text{ ramifies,}\\ 0\ \mathbb{F}_q\text{-points,} & \text{if } v \text{ is inert.}\end{cases}$$

More generally, a place v of K corresponds to a Galois orbit of rational points of the same degree of $\mathbb{P}^1$. The fiber above each point in the orbit contains

$$\begin{cases} \ell \text{ distinct points of degree } \deg v & \text{if } v \text{ splits completely,}\\ 1 \text{ point of degree } \deg v & \text{if } v \text{ ramifies,}\\ 1 \text{ point of degree } \ell\deg v & \text{if } v \text{ is inert.}\end{cases}$$

To get the distribution of $\#C(\mathbb{F}_q)$ over $\mathcal{H}_{g,\ell}$, we use the relative densities

$$\frac{N(\mathbb{Z}/\ell\mathbb{Z}, \frac{2g}{\ell-1}+2; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)}{N(\mathbb{Z}/\ell\mathbb{Z}, \frac{2g}{\ell-1}+2)}$$

where we take the sets $\mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I$ to be mutually disjoint, and such that $\mathcal{V}_R \cup \mathcal{V}_S \cup \mathcal{V}_I$ is a subset of the set of primes of degree 1 in $\mathcal{V}_K$.

Then, using (26) with $n = 1$ and Theorem 1.1, we get

$$\begin{aligned}
&\frac{|\{C \in \mathcal{H}_{g,\ell}(\mathbb{F}_q) \ : \ \#C(\mathbb{F}_q) = m\}|}{|\mathcal{H}_{g,\ell}(\mathbb{F}_q)|} \\
&= \sum_{\ell|\mathcal{V}_S|+|\mathcal{V}_R|=m} \frac{N(\mathbb{Z}/\ell\mathbb{Z}, \frac{2g}{\ell-1}+2; \mathcal{V}_R, \mathcal{V}_S, \mathcal{V}_I)}{N(\mathbb{Z}/\ell\mathbb{Z}, \frac{2g}{\ell-1}+2)} \\
\sim \ & \sum_{\ell|\mathcal{V}_S|+|\mathcal{V}_R|=m} \left(\frac{\ell-1}{q+\ell-1}\right)^{|\mathcal{V}_R|} \left(\frac{q}{\ell(q+\ell-1)}\right)^{|\mathcal{V}_S|} \left(\frac{(\ell-1)q}{\ell(q+\ell-1)}\right)^{q+1-|\mathcal{V}_R|-|\mathcal{V}_R|} \\
= \ & \operatorname{Prob}\left(\sum_{i=1}^{q+1} X_i = m\right),
\end{aligned}$$

where the X_i are the random variables of Theorem 1.3.

5.1. **Affine models.** We compare the results of this paper with the results of [BDFL10] concerning the irreducible components $\mathcal{H}^{(d_1,\dots,d_\ell)}$ of $\mathcal{H}_{g,\ell}$. To describe those components, we write the covers concretely in terms of affine models. Each such cover has an affine model of the form

$$C : Y^\ell = f(X) = \beta f_1 f_2^2 \cdots f_{\ell-1}^{\ell-1} \tag{27}$$

where the $f_i \in \mathbb{F}_q[X]$ are monic, square-free, and coprime in pairs, of degrees $d_1, \dots, d_{\ell-1}$. The degree of the conductor depends on the degrees $d_1, \dots, d_{\ell-1}$ and whether there is ramification at the infinite place. The ramification at the infinite place is determined by whether the total degree of the polynomial is divisible by ℓ. When $d_1 + \cdots + (\ell-1)d_{\ell-1}$ is a multiple of ℓ, then the cover does not ramify at infinity, otherwise there is ramification at infinity. In the first case the degree of the conductor is $d_1+\cdots+d_{\ell-1}$ and in the second case it is $d_1+\cdots+d_{\ell-1}+1$.

By the Riemann-Hurwitz formula the genus of this cover is given by

$$g_C = (\ell-1)(d_1 + \cdots + d_{\ell-1} - 2)/2 = (\ell-1)(\mathfrak{n}-2)/2$$

in the first case, and

$$g_C = (\ell-1)(d_1 + \cdots + d_{\ell-1} - 1)/2 = (\ell-1)(\mathfrak{n}-2)/2,$$

in the second. Either way, we recover the relation (23) between the genus g and the degree of the conductor $\mathfrak{n}$.

For a given conductor, each $\beta \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^\ell$ yields a different cover given by formula (27). That is, there is one such extension for each element of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^\ell$. Using the notation from [BDFL10], we define

$$\mathcal{F}_{(d_1,\dots,d_{\ell-1})} = \{(f_1, \dots, f_{\ell-1}) : f_i \text{ monic, square-free, pairwise coprime, } \deg f_i = d_i, i = 1, \dots, \ell-1\}.$$

Formula (3.1) of [BDFL10] says that

$$|\mathcal{F}_{(d_1,\dots,d_{\ell-1})}| = \frac{L_{\ell-2} q^{d_1+\cdots+d_{\ell-1}}}{\zeta_q(2)^{\ell-1}} \left(1 + O\left(\sum_{h=2}^{\ell-1} q^{\varepsilon(d_h+\cdots+d_{\ell-1})-d_h} + q^{-d_1/2}\right)\right), \tag{28}$$

where

$$L_{\ell-2} = \prod_{j=1}^{\ell-2} \prod_{v \neq \infty} \left(1 - \frac{jq^{-2\deg v}}{(1+q^{-\deg v})(1+jq^{-\deg v})}\right).$$

The formula above may be rewritten as

$$|\mathcal{F}_{(d_1,\ldots,d_{\ell-1})}| = \frac{(\ell-2)! C_\ell q^{d_1+\cdots+d_{\ell-1}}}{(1+(\ell-1)q^{-1})} \left(1 + O\left(\sum_{h=2}^{\ell-1} q^{\varepsilon(d_h+\cdots+d_{\ell-1})-d_h} + q^{-d_1/2}\right)\right).$$

Now suppose that we want to count the number of covers of genus g. For a conductor $f_1 f_2^2 \cdots f_{\ell-1}^{\ell-1}$, there are ℓ different covers according to the class of leading coefficient as an element of $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^\ell$. Thus, we can write

$$|\mathcal{H}_{g,\ell}(\mathbb{F}_q)| = \sum_{\substack{d_1+\cdots+d_{\ell-1}=2(g+2)/(\ell-1) \\ d_1+\cdots+(\ell-1)d_{\ell-1}\equiv 0 \bmod \ell}} \ell |\mathcal{F}_{[d_1,\ldots,d_{\ell-1}]}|, \tag{29}$$

where, for $d_1 + \cdots + (\ell-1)d_{\ell-1} \equiv 0 \pmod{\ell}$,

$$\mathcal{F}_{[d_1,\ldots,d_{\ell-1}]} = \mathcal{F}_{(d_1,\ldots,d_{\ell-1})} \cup \bigcup_{j=1}^{\ell-1} \mathcal{F}_{(d_1,\ldots,d_j-1,\ldots,d_{\ell-1})}.$$

This gives

$$\begin{aligned} |\mathcal{H}_{g,\ell}(\mathbb{F}_q)| &= (\ell-2)! C_\ell q^{\mathfrak{n}} \sum_{\substack{d_1+\cdots+d_{\ell-1}=\mathfrak{n} \\ d_1+\cdots+(\ell-1)d_{\ell-1}\equiv 0 \bmod \ell}} \ell + ET \\ &= C_\ell \mathfrak{n}^{\ell-2} q^{\mathfrak{n}} + ET, \end{aligned}$$

where ET denotes an error term and in the last line, we used that the number of solutions of $d_1 + \cdots + d_{\ell-1} = \mathfrak{n}$ is given by $\binom{\mathfrak{n}+\ell-2}{\mathfrak{n}} \sim \frac{\mathfrak{n}^{\ell-2}}{(\ell-2)!}$.

Then, the result of Theorem 4.1 is compatible with the result of Theorem 3.1 from [BDFL10], in the sense that the summing the main terms of (28) gives the number of elements of $\mathcal{H}_{g,\ell}$ as computed with the techniques of this paper, even if the error terms coming from (28) are only valid when $d_1, \ldots, d_{\ell-1} \to \infty$.

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