Harmonic Frames of Prime Order

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Outline

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Finite frames

- Let $\Phi = \{\varphi_i\}_{i=1}^{s} \subset \mathbb{C}^d$.

Definition

$\Phi = \{\varphi_i\}_{i=1}^{s}$ is a *finite frame* for $\mathbb{C}^d$ if there exists $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{s} |\langle f, \varphi_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathbb{C}^d.$$

- If $\|\varphi_i\| = 1$ for all $i = 1, \ldots, s$, then Φ is a *unit norm frame*.
- If we can take $A = B$ in the definition, then Φ is a *tight frame*.
- If Φ is unit norm and tight, then Φ is a *finite unit norm tight frame*, or FUNTF, and we can take $A = B = s/d$.
- Φ is a frame for $\mathbb{C}^d$ if and only if $\mathrm{span}(\Phi) = \mathbb{C}^d$.

Outline

DFT-FUNTFs

- The $s \times s$ inverse Discrete Fourier Transform (DFT) matrix is given by:

$$D_s = (e^{2\pi imn/s})_{n,m=0}^{s-1}.$$

- Define the group of integers mod s as:

$$\mathbb{Z}_s = \mathbb{Z}/s\mathbb{Z} = \{0, \ldots, s-1 \mod s\}.$$

Definition

Choose $d \leq s$ distinct rows of D_s, say $n = (n_1, \ldots, n_d) \in \mathbb{Z}_s^d$, and define vectors φ_m as

$$\varphi_m = \frac{1}{\sqrt{d}}(e^{2\pi imn_j/s})_{j=1}^{d}, \quad \forall m \in \mathbb{Z}_s.$$

$\Phi_n = \{\varphi_m : m \in \mathbb{Z}_s\}$ is a FUNTF for $\mathbb{C}^d$, and any frame of this type is called a *DFT-FUNTF*. Call $n = (n_1, \ldots, n_d)$ the *generators* of Φ_n.

An example

- Let $s = 4$ and $d = 3$.
- $$D_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

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An example

- Let $s = 4$ and $d = 3$.
-

$$\begin{array}{cccc} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$$

$$D_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ & & & \end{pmatrix} \begin{array}{l} \leftarrow n_1 = 0 \\ \leftarrow n_2 = 1 \\ \leftarrow n_3 = 2 \\ \end{array}$$

A simple question?

- A seemingly simple homework question: for a fixed s (the number of frame elements) and d (the dimension of the space) count the number of DFT-FUNTFs.
- Initial guesses:
  - Order of row choices matters:

$$s(s-1)(s-2)\cdots(s-d+1).$$

  - Order of row choices does not matter:

$$\binom{s}{d}.$$

- Unfortunately neither is quite correct...

Order does not matter?

- Let $s = 3$ and $d = 2$.
-
$$D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{matrix} \leftarrow \text{row } 0 \\ \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \end{matrix}$$

- Let $n_1 = 1$ and $n_2 = 2$. Then:

$$\varphi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}.$$

- Now let $n_1' = 2$ and $n_2' = 1$. Then:

$$\varphi_0' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1' = \begin{pmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}, \quad \varphi_2' = \begin{pmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix}.$$

- They are the same frame!

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Order does not matter?

- Let $s = 3$ and $d = 2$.
-
$$D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{matrix} \leftarrow \text{row } 0 \\ \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \end{matrix}$$
- Let $n_1 = 1$ and $n_2 = 2$. Then:
$$\varphi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}.$$
- Now let $n_1' = 2$ and $n_2' = 1$. Then:
$$\varphi_0' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1' = \begin{pmatrix} e^{4\pi i/3} \\ e^{2\pi i/3} \end{pmatrix}, \quad \varphi_2' = \begin{pmatrix} e^{2\pi i/3} \\ e^{4\pi i/3} \end{pmatrix}.$$
- They are the same frame!

On the other hand...

- Let $s = 3$ and $d = 2$.
- $$D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{matrix} \leftarrow \text{row } 0 \\ \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \end{matrix}$$
- Let $n_1 = 0$ and $n_2 = 1$. Then:
$$\varphi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 1 \\ e^{2\pi i/3} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 \\ e^{4\pi i/3} \end{pmatrix}.$$
- Now let $n_1' = 1$ and $n_2' = 0$. Then:
$$\varphi_0' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1' = \begin{pmatrix} e^{2\pi i/3} \\ 1 \end{pmatrix}, \quad \varphi_2' = \begin{pmatrix} e^{4\pi i/3} \\ 1 \end{pmatrix}.$$
- They are not quite the same frame... but almost.

Almost the same

- In the previous example we had the two DFT-FUNTFs:

$$\begin{aligned}\Phi_{(0,1)} &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ e^{2\pi i/3} \end{pmatrix}, \begin{pmatrix} 1 \\ e^{4\pi i/3} \end{pmatrix} \right\}, \\ \Phi_{(1,0)} &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} e^{2\pi i/3} \\ 1 \end{pmatrix}, \begin{pmatrix} e^{4\pi i/3} \\ 1 \end{pmatrix} \right\}.\end{aligned}$$

- Notice a simple permutation of the 'rows' of $\Phi_{(1,0)}$ in fact gives us $\Phi_{(0,1)}$. Thus, at the very least, they are quite similar.
- Running with this observation, and inspired by the previous two examples, we put the following equivalence relation on the DFT-FUNTFs...

An equivalence relation for DFT-FUNTFs

- For $k \in \mathbb{Z}$, $k \geq 1$, let S_k denote the group of permutations on k elements.

Definition

Let $\Phi_n = \{\varphi_m : m \in \mathbb{Z}_s\} \subset \mathbb{C}^d$ and $\Phi_{n'} = \{\varphi'_m : m \in \mathbb{Z}_s\} \subset \mathbb{C}^d$ be DFT-FUNTFs. Φ_n and $\Phi_{n'}$ are equivalent if and only if there exists $\sigma_1 \in S_s$ and $\sigma_2 \in S_d$ such that

$$\varphi_m(k) = \varphi'_{\sigma_1(m)}(\sigma_2(k)),$$

for all $m \in \mathbb{Z}_s$ and for all $k = 1, \ldots, d$.

- σ_1 just says the indexing of the frame elements does not matter.
- σ_2 is the 'row' permutation.

A natural question

- Is the number of equivalence classes of DFT-FUNTFs now $\binom{s}{d}$?
- Unfortunately, no.

A natural question

- Is the number of equivalence classes of DFT-FUNTFs now $\binom{s}{d}$?
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One more example

- Let $s = 3$ and $d = 2$.
- $$D_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{pmatrix} \begin{matrix} \leftarrow \text{row } 0 \\ \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \end{matrix}$$
- Let $n_1 = 0$ and $n_2 = 2$. Then:
$$\varphi_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 1 \\ e^{4\pi i/3} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 1 \\ e^{2\pi i/3} \end{pmatrix}.$$
- Thus, $\Phi_{(2,0)}$ and $\Phi_{(1,0)}$ are the same frame, and so both are also in the same equivalence class of $\Phi_{(0,1)}$.

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So what does it mean?

- So what does this equivalence relation mean?
- For the answer we turn to harmonic frames...

Outline

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Characters

- Harmonic frames are a generalization of DFT-FUNTFs.
- Let $G = \{g_i\}_{i=1}^s$ denote a finite abelian group.

Definition

A *character* of a finite abelian group G is a group homomorphism $\xi : G \longrightarrow \mathbb{C}^\times$, i.e.

$$\xi(g_i g_j) = \xi(g_i)\xi(g_j), \quad \forall\, g_i, g_j \in G.$$

- For each $g_i \in G$, $\xi(g_i)$ is a s-th root of unity.
- There are exactly s characters, $\{\xi_i\}_{i=1}^s$.
- The *character table* of G is the matrix $(\xi_i(g_j))_{i,j=1}^s$.
- When $G \cong \mathbb{Z}_s$, the character table of G is D_s.

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Harmonic frames

- Let $\mathcal{U}(\mathbb{C}^d)$ denote the group of unitary transformations (matrices) on $\mathbb{C}^d$:

$$\mathcal{U}(\mathbb{C}^d) = \{U \in \mathcal{M}_{d\times d}(\mathbb{C}) : U^\star U = UU^\star = I_{d\times d}\}.$$

Definition

Let $G = \{g_i\}_{i=1}^s$ be a finite abelian group and let $\mathcal{J} \subseteq \{1, \ldots, s\}$ such that $|\mathcal{J}| = d$. Define vectors $\tilde{\varphi}_{g_i} \in \mathbb{C}^d$ as

$$\tilde{\varphi}_{g_i} = (\xi_j(g_i))_{j\in\mathcal{J}}, \quad \forall\, g_i \in G.$$

Then for any $U \in \mathcal{U}(\mathbb{C}^d)$ the set

$$\Phi = \{\varphi_i\}_{i=1}^s = \{U\tilde{\varphi}_{g_i} : g_i \in G\} \subset \mathbb{C}^d,$$

is a frame for $\mathbb{C}^d$ and is called a *harmonic frame*.

Equivalence classes of harmonic frames

- Since there are an infinite number of harmonic frames, how do we count them?

Definition

Two harmonic frames $\Phi = \{\varphi_i\}_{i=1}^{s} \subset \mathbb{C}^d$ and $\Phi' = \{\varphi_i'\}_{i=1}^{s} \subset \mathbb{C}^d$ are said to be *equivalent* if there exists $U \in \mathcal{U}(\mathbb{C}^d)$ such that

$$\{\varphi_i\}_{i=1}^{s} = \{U\varphi_i'\}_{i=1}^{s}.$$

We denote this equivalence relation as $\Phi \sim \Phi'$.

- But what does this have to do with our previous equivalence relation on DFT-FUNTFs?

The first link

- When s is prime, the only finite abelian group G with s elements is $\mathbb{Z}_s$.
- Therefore, when s is prime, every harmonic frame is derived from the character table of $\mathbb{Z}_s$, which is D_s.
- Thus, every prime order harmonic frame is of the form $U\Phi_n$, where $U \in \mathcal{U}(\mathbb{C}^d)$ and Φ_n is a DFT-FUNTF.

Remark

Combining the above statements, we see that if one wants to count the number of equivalence classes of prime order harmonic frames, one only needs to count the number of inequivalent DFT-FUNTFs under $\sim$.

The second link

- The following lemma ties everything together.

Lemma

Let s be a prime number. If $\Phi_n = \{\varphi_m : m \in \mathbb{Z}_s\}$ and $\Phi_{n'} = \{\varphi'_m : m \in \mathbb{Z}_s\}$ are DFT-FUNTFs, then

$$\begin{array}{c} \exists\ U \in \mathcal{U}(\mathbb{C}^d) \\ \text{such that} \\ \Phi_n = U\Phi_{n'} \end{array} \iff \begin{array}{c} \exists\ \sigma_1 \in S_s,\ \ \sigma_2 \in S_d \text{ such that} \\ \varphi_m(k) = \varphi'_{\sigma_1(m)}(\sigma_2(k)) \\ \forall\ m \in \mathbb{Z}_s,\ \ k = 1, \ldots, d. \end{array}$$

- Thus when s is a prime number the two equivalence relations are in fact the same, and so not only is the right hand side given a precise meaning, it is a useful tool for counting all equivalence classes of harmonic frames of prime order.

Overview of the enumeration problem

- **Goal:** Enumerate inequivalent harmonic frames, i.e. count the number of equivalence classes.
- **Results:** Exact, recursive formula for the number of inequivalent, prime order harmonic frames (i.e. when s is prime).
- The prime order case is simpler than the general case in part because there is only one prime order abelian group, namely $\mathbb{Z}_s$.
- This work builds upon results by:
  - Vale and Waldron (2005) - developed harmonic frames.
  - Hay and Waldron (2006) - wrote a computer program that computes all harmonic frames for a given s and d; conjectured that the number of inequivalent harmonic frames is $\mathcal{O}(s^{d-1})$.

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Setup for the main result

- For a fixed s (prime) and d ($d \leq s$), we backwards recursively define the set:

$$\{\gamma_c \in \mathbb{N} \cup \{0\} : c \in \mathbb{N}, c \mid s-1, \text{ and } c \mid d \text{ or } c \mid d-1\}.$$

- If $c \mid s-1$, $c \mid d$, and $c > 1$, then:

$$\gamma_c = \frac{(s-1-c)(s-1-2c)\cdots(s-1-(\frac{d}{c}-1)c)}{c^{\frac{d}{c}-1}(d/c)!} - \frac{c}{s-1} \sum_{\substack{c<b<s \\ c|b,\, b|d}} \binom{s-1}{b} \gamma_b.$$

- If $c \mid s-1$, $c \mid d-1$, and $c > 1$ then we define γ_c similarly - simply replace d with $d-1$.
- For $c = 1$, define

$$\gamma_1 = \frac{1}{s-1}\binom{s}{d} - \sum_{\substack{c|d \\ c>1}} \frac{\gamma_c}{c} - \sum_{\substack{c|d-1 \\ c>1}} \frac{\gamma_c}{c}.$$

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The main result

Theorem

Let s be a prime number and let $1 < d < s$. Define constants γ_c as in the previous slide. The total number of inequivalent harmonic frames for $\mathbb{C}^d$ with s elements is given by:

$$\gamma_1 + \sum_{\substack{c|d \\ c>1}} \gamma_c + \sum_{\substack{c|d-1 \\ c>1}} \gamma_c.$$

Corollary

Let s be any prime number and fix d such that $1 < d < s$. Then the number of inequivalent harmonic frames for $\mathbb{C}^d$ with s elements is $\mathcal{O}(s^{d-1})$.

Outline

Back to the original homework problem

- Recall how this all began: for a fixed s (the number of frame elements) and d (the dimension of the space) count the number of DFT-FUNTFs.
- When s is prime, the solution to this problem turned out to be much simpler...

Proposition

Let s be a prime number and let $d \leq s$. The number of DFT-FUNTFs with s elements for $\mathbb{C}^d$ is given by

$$s(s-2)(s-3)\cdots(s-d+1).$$

Thank you

Thank you!