PRIME CONGRUENCES OF IDEMPOTENT SEMIRINGS
AND A NULLSTELLENSATZ FOR TROPICAL POLYNOMIALS

SUMMARY

- We give a new definition of prime congruences in additively idempotent semirings. These congruences have analogous properties to the prime ideals of commutative rings.
- A complete description of prime congruences is given in the polynomial and Laurent polynomial semirings over the tropical semifield \( \mathbb{R}^{\max} \) and the semifields \( \mathbb{Z}^{\max} \) and \( \mathbb{B} \).
- The minimal primes of these semirings correspond to monomial orderings, and their intersection is the congruence that identifies polynomials that have the same Newton polytope.
- The Krull dimension of the (Laurent) polynomial semiring in \( n \) variables over \( K \) (where \( K \) one the three studied semifields above) is equal to \( \dim(K) + n \).
- The radical of every finitely generated congruence in the studied cases is the intersection of prime congruences with quotients of dimension 1.
- An improvement of a result by A. Bertram and R. Easton is proven which can be regarded as a Nullstellensatz for tropical polynomials.

CONGRUENCES OF IDEMPOTENT SEMIRINGS

**Motivation:** For the traditional tropical geometry (e.g. Sturmfels, MacLagan, Mikhalkin) a tropical variety (over \( \mathbb{R}^{\max} \)) is a commutative monoid with respect to both (usual addition, with \( \max \) is just the subsemifield of integers in \( \mathbb{R}^{\max} \)).

**Definition:** A semiring is the ordered pair \( (R, +, \times) \) such that \( R \) is a commutative monoid with respect to both (usual addition and multiplication the usual addition being the usual maximum and multiplication the usual \( \times \)).

- All nonzero elements have multiplicative inverse. Examples are:
  - \( \mathbb{B} \) the semifield with two elements \( \{0, 1\} \).
  - \( \mathbb{Z}^{\max} \) the semifield with underlying set \( \{\infty, 0\} \cup \mathbb{R} \) addition being the usual maximum and multiplication the usual addition, with \( -\infty \) playing the role of the 0 element.
  - \( \mathbb{R}^{\max} \) is just the subsemifield of integers in \( \mathbb{R}^{\max} \).

PRIME CONGRUENCES OF IDEMPOTENT SEMIRINGS

- Quotients by a prime are totally ordered with respect to the ordering coming from the idempotent addition.
- For a prime \( P \) of \( \mathbb{B} \) \( \mathbb{R}^{\max} \) the multiplicative monoid of \( \mathbb{B}(x)/P \) \( \mathbb{R}^{\max}(x)/P \) is isomorphic to a quotient of the additive group \( \mathbb{Z}^2 \) resp. to the restiction of a quotient \( \mathbb{Z}^n \) to \( \mathbb{N}^n \) where \( n = n - n = (x_1, \ldots, x_n) \cap \text{Ker}(P)) \).
- To understand the prime quotients of \( \mathbb{B}(x) \) we need to describe the group orderings on the quotients of \( \mathbb{Z}^2 \).

Criteria: These orderings can be given by a defining matrix \( U \) so that \( m > n \) only if \( U_m > U_n \) with respect to lex order. We denote by \( P(U) \) the prime in \( \mathbb{B}(x) \) corresponding to the ordering given by \( U \).

**Definition:** The dimension of a \( \mathbb{B} \)-algebra \( A \) is the length of the longest chain (with respect to inclusion) of prime congruences in \( A \).

**Theorem:** For a congruence \( I \) of a \( \mathbb{B} \)-algebra \( A \), \( \text{Rad}(I) = \langle \alpha \mid \text{GP}(\alpha) \cap I \neq \emptyset \rangle \).

- Every prime congruence \( P \) of \( \mathbb{B}(x) \) with trivial kernel is of the form \( P(U) \mathbb{B}(x) \).
- \( \text{dim}(\mathbb{B}(x)/P(U)) = \text{rank } U \) and \( \text{dim}(\mathbb{B}(x)/P(U)\mathbb{B}(x)) = \text{rank } U \) in particular \( \text{dim}(\mathbb{B}(x)) = \text{dim}(\mathbb{B}(x)) = n \).

**Theorem:** The pair \((f, g)\) lies in the radical of \( \Delta \) of \( \mathbb{B}(x) \) or \( \mathbb{B}(x) \) if and only if the Newton polytopes of \( f \) and \( g \) are the same.

**Definition:** If \( \text{dim}(\mathbb{B}(x)) = n \) the \( \mathbb{B}(x)/\text{Rad}(\Delta) \) is isomorphic to the \( \mathbb{B}(x) \)-algebra with elements the lattice polytopes and addition being defined as the convex hull of the union, and multiplication as the Minkowski sum.

Analogous results hold over the \( \mathbb{B}(x) \)-algebra \( \mathbb{R}^{\max}(x)/P(U) \) and instead the of Newton polytope \( \text{newt}(f) \) consider, \( \text{newt}(f) = \{\{0, 0, \ldots, y_k\} \in \text{newt}(f) \mid \forall z > y : \{z, y_1, \ldots, y_k\} \notin \text{newt}(f)\} \).

Those semialgebras have dimensions \( n + 1 \), where \( n \) is the number of variables.

TROPICAL NULLSTELLENSATZ

**Motivation:** We are interested in subsets of \( \mathbb{B}(x) \) where some finite collection \( (f_i, g_i) \) of pairs of tropical polynomials agree, i.e., \( \{f_i \mathbb{B}(x) = g_i \mathbb{B}(x)\} \).

**Theorem:** When \( E \) is finitely generated \( E \) is the intersection of all geometric congruences containing \( E \), in particular \( E \) is a congruence and \( V(E) = V(E) \).

**Theorem:** For a finitely generated congruence \( E \) in the (Laurent) polynomial semiring over \( \mathbb{B}, \mathbb{R}^{\max} \) or \( \mathbb{R}^{\max} \), \( \text{Rad}(E) \) is the intersection of the primes that contain \( E \) and have a quotient with dimension 1.