Review for MT 2

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- Matrices & matrix operations
- Write linear systems with matrices

$$\left.\begin{array}{l} x+y+2z=7 \\ 3x \quad -7z=9 \end{array}\right\} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 0 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \Bigg\} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 3 & 0 & -7 & 9 \end{array}\right] \Bigg\} \text{Gaussi Elimn.}$$

Note: find inverse, go all the way to reduced echelon form.

The inverse of a square matrix A,

Inverse A^{-1} exists : $A \cdot A^{-1} = I$, $A^{-1} \cdot A = I$

The inverse of A exists $\det A \neq 0$

① $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -9 \\ 3 & 4 & 5 \end{bmatrix}$ $C^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -2 & 0 & 6 \end{bmatrix}$

a). is it invertible?

$\det A = 1(5)(9) + (4)(8)(3) + \cdots = 0$ so A is not invertible.

b). How many solutions does $Ax = 0$ have? Inf

- Infinitely many solutions because A is not invertible.

c). What is det B? [Use method for large matrices]

$$\det B = \det\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -9 \\ 3 & 4 & 5 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 4 & 5 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 0 & -2 & -4 \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 0 & -6 \end{bmatrix} = -(1)(-2)(-6) = -12$$

d). What is det C? [1st find $\det C^{-1}$]

$$\det C^{-1} = \det\left[\begin{array}{c|c|c} 0 & 1 & 0 \\ 3 & 0 & 1 \\ -2 & 0 & 6 \end{array}\right] = -1 \det\begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix} + 0 - 0$$

$$= -[\underbrace{3(6) - (-2)(1)}_{18 + 2}] = -20$$

$A^{-1}(Ax = (b)^{-1}$

Continued c). $C \cdot C^{-1} = I$

$\det(C \cdot C^{-1}) = \det I$

$\det C \cdot \det C^{-1} = \det I \rightarrow \det C \cdot (-20) = 1$

$\det C \cdot = \frac{-1}{20}$

e). What is $\det((BC)^T)$?

[Remember $\det M^T = \det M$ & $(MN)^T = N^T M^T$]

$\det(BC) = \det B \cdot \det C = (-12)(\frac{-1}{20}) = 3/5$

$\det((BC)^T) = \det(C^T B^T)$

$= \det C^T \cdot \det B^T$

$= \det C \cdot \det B = (\frac{-1}{20})(-12) = 3/5$

f). Remember $\mathbb{R}^3$ means 3D space.

- The matrix B represents some linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

" C " $\mathbb{R}^3$ to $\mathbb{R}^3$.

- Find the matrix that represents the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "C"] & \mathbb{R}^3 \arrow[r, "B"] & \mathbb{R}^3 \\
x \arrow[r] & Cx \arrow[r] & BCx
\end{tikzcd}$$

BC represents this linear transformation.

Need to find BC.

- first find C, $(C^{-1})^{-1} = C$

$$\underbrace{\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ -2 & 0 & 6 & 0 & 0 & 1 \end{array}\right]}_{C^{-1}} \rightarrow \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \quad ① = ① + ③$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{20}{20} & 0 & 2 & 3 \end{array}\right] \rightarrow$$

$$③ = 2① + ③ \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/10 & 3/20 \end{array}\right] \xrightarrow{① = ① - 7③} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3/10 & -1/20 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/10 & 3/20 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 3/10 & -1/20 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/10 & 3/20 \end{array}\right] \longrightarrow \text{Then } C = \frac{1}{20}\begin{bmatrix} 0 & 6 & -1 \\ 20 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

$$(C^{-1})^{-1} = C$$

Finally: $\underline{BC} = \frac{1}{20}\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -9 \\ 3 & 4 & 5 \end{bmatrix}\begin{bmatrix} 0 & 6 & -1 \\ 20 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$ $1-27$

$$BC = \frac{1}{20}\begin{bmatrix} 40 & 12 & 8 \\ -40 & -24 & -26 \\ 80 & 28 & 12 \end{bmatrix}$$

② Find the eigenvalues and eigenvectors of rotation by 90° C.W in 2D.

– Call R this rotation (and matrix that represents it)

$$R = \begin{bmatrix} Re_1 & Re_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$Re_1 = R\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad Re_2 = R\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

? λ replace 0 ?

$$\det(R - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

Now solve $\lambda^2 + 1 = 0$ $\quad (\lambda - i)(\lambda + i) = 0$

$\lambda - i = 0$ or $\lambda + i = 0$

$\boxed{\lambda = i}$ $\quad \boxed{\lambda = -i}$

[Note: complex numbers, or use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ instead of factoring]

These are eigenvalues of R.

Eigenvectors for $\lambda = i$: Solve $(R - iI)x = 0$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{So} \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \longrightarrow \begin{array}{cc} \text{basic} & \text{free} \\ 1 & i \\ \left[\begin{matrix} \boxed{-i} \\ 0 \end{matrix}\right. & \left.\begin{matrix} 1 \\ 0 \end{matrix}\right] \end{array}$$

} If done correctly, get row of zeros in REF.

$-ix_1 + x_2 = 0 \rightarrow -i \cdot i = -i x_2 \rightarrow -(-1)x_1 = -ix_2$

$-ix_1 = -x_2 \rightarrow x_1 = -ix_2$ $\qquad \boxed{x_1 = -ix_2}$

$\frac{i^2 x_2}{-i}$

$= -ix_2$

$x =$

Eigenvectors for $\lambda = -i$

get $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$

b). Without multiplying find

$R\begin{bmatrix} i \\ 1 \end{bmatrix} = -i\begin{bmatrix} i \\ 1 \end{bmatrix}$ } This is the definition

③ Random Walks
④ Linear transformation (def & matrix representation)
⑤ Resistor Networks

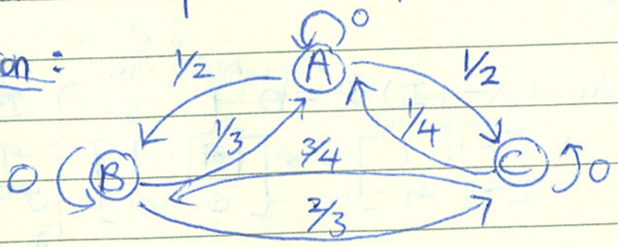
③ The country of Dulland is extremely boring. It has only 3 cities A, B, and C place. Every year each inhabitant gets bored and decides to move to a different city in Dulland.

- The prob a person living in A moves to B is ½.
  B A is ⅓.
  C A is ¼.

a). If a person lives in A in 2012, where is the person most likely to live in 2014?

b). What is the prob that a person lives in A in 2012 lives in C in 2016?

Solution :



A = state 1
B = state 2
C = state 3

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P_{ij} = prob of moving from j to i.

The matrix representing the random walks is

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/4 \\ 1/2 & 0 & 3/4 \\ 1/2 & 2/3 & 0 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 0 & 4 & 3 \\ 6 & 0 & 9 \\ 6 & 8 & 0 \end{bmatrix}$$

[Note: Column must add up to 1]

$$P^2 = P \cdot P = \frac{1}{12^2}\begin{bmatrix} \boxed{42} & 24 & 36 \\ \boxed{54} & 96 & 18 \\ \boxed{48} & 24 & 90 \end{bmatrix} \qquad \underset{\substack{\downarrow \\ P^2 \cdot P^2}}{P^4} = \frac{1}{12^4}\begin{bmatrix} x & x & x \\ x & x & x \\ 7632 & x & x \end{bmatrix}$$

a). In 2014:

prob A to A: $\frac{42}{144}$, prob A to B: $\frac{54}{144}$, prob A to C: $\frac{48}{144}$

- The person is <u>most likely</u> to be living in B in 2014!

b). The prob living in A → C in 2016:

$$\frac{7632}{12^4} \approx 0.3757 \approx 35.57\%$$

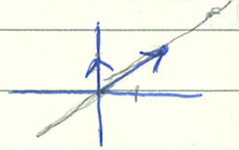
Eigen values & Eigenvectors [Note: A is 2X2 matrice that's unknown]

a). A $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ is eigenvector corresp to eigen value 3 of A.

$$A \cdot \begin{bmatrix} 10 \\ 20 \end{bmatrix} = 3 \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \end{bmatrix} \qquad B = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

b). $\underbrace{B \begin{bmatrix} 5 \\ 7 \end{bmatrix}}_{\uparrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

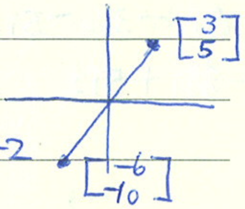
Not Eigenvector



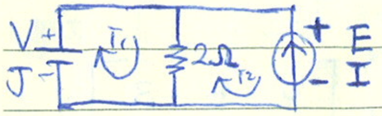
$\underset{\uparrow}{B \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}} = \begin{bmatrix} -6 \\ -10 \end{bmatrix}$

Eigenvector for eigenvalue $\lambda = -2$

$$\begin{bmatrix} -6 \\ -10 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$



⑤ Resistor Networks



Given Voltage V & Current I

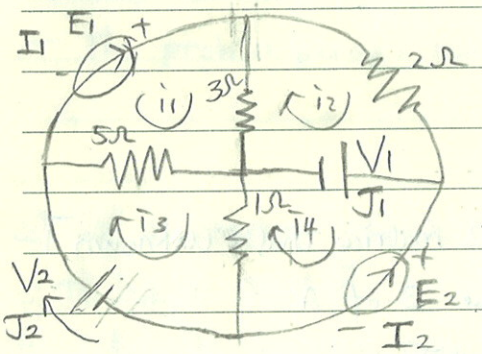
Express voltage E & Current J in terms of V & I.

Loop Currents : unknown i_1, i_2, E.

$$\left.\begin{aligned} 2(i_1-i_2)-V&=0\\ 2(i_2-i_1)+E&=0\\ -i_2&=I\end{aligned}\right\} \rightarrow \begin{aligned} 2i_1-2i_2&=V \rightarrow 2i_1+2I=V\\ 2i_2-2i_1+E&=0\\ -i_2&=I\end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & V\\ -2 & 2 & 1 & 0\\ 0 & -1 & 0 & I\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & \frac{1}{2}V\\ 0 & 0 & 1 & V\\ 0 & 1 & 0 & -I\end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & \enclose{circle}{-1} & 0 & \frac{1}{2}V\\ 0 & 1 & 0 & -I\\ 0 & 0 & 1 & V\end{array}\right]$$

$$\begin{aligned} i_1-i_2&=\tfrac{1}{2}V\\ i_2&=-I\\ E&=V\end{aligned} \quad\uparrow\quad \begin{aligned} E&=V\\ i_1+I&=\tfrac{1}{2}V\\ i_1&=\tfrac{1}{2}V-I\end{aligned} \quad\nearrow\quad \begin{aligned} E&=V\\ J=i_1&=\tfrac{1}{2}V-I\end{aligned}$$



Loop Currents : Unknown $i_1, i_2, i_3, i_4, E, E_2$

$3(i_1-i_2)+5(i_1-i_3)-E_1=0$

$2(i_2)+V_1+3(i_2-i_1)=0$

$i_1 = I_1$

$i_4 = -I_2$

$8i_1 - 3i_2 - 5i_3 - E_1 = 0$

$-3i_1 + 5i_2 = -V_1$

Final step: $J_1 = i_4 - i_2$

$J_2 = i_3$

④ Linear Transformation

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① $T: R^2 \to R^3$ $\quad T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x-2y\\2x-3y\\x^2-x\end{bmatrix}$ Is T Linear Transformation?

$$\left.\begin{array}{l}\underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}\right)}_{T\begin{bmatrix}3\\0\end{bmatrix}} \overset{?}{=} \underbrace{T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)} + \underbrace{T\left(\begin{bmatrix}2\\0\end{bmatrix}\right)} \\ \qquad\qquad\qquad\qquad = \begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}2\\4\\2\end{bmatrix} \\ = \begin{bmatrix}3\\6\\6\end{bmatrix} \xleftrightarrow[\text{equal}]{\text{do not}} \begin{bmatrix}3\\6\\2\end{bmatrix}\end{array}\right\} \text{Compatible } \overline{w} \text{ sum?}$$

Ans: No ! because

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}2\\0\end{bmatrix}\right) \overset{?}{=} T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+T\left(\begin{bmatrix}2\\0\end{bmatrix}\right)$$

Compatible with scalar?

$$\left.\begin{array}{l}T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right) \overset{?}{=} 2T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \\ = \begin{bmatrix}2\\4\\2\end{bmatrix} \qquad 2\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\4\\0\end{bmatrix}\end{array}\right\} \text{Ans: No!}$$

Summary: $f(a+b) = f(a) + f(b)$

$f(t \cdot a) = t\, f(a)$

② $T: R^2 \to R^3$ $\quad T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x-2y\\2x-3y\\-x\end{bmatrix}$

① Compatible $\overline{w}$ sum:

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}\right) \overset{?}{=} T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)+T\left(\begin{bmatrix}c\\d\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}\right) = \begin{bmatrix}(a+c)-2(b+d)\\2(a+c)-3(b+d)\\-(a+c)\end{bmatrix} = \begin{bmatrix}a+c-2b-2d\\2a+2c-3b-3d\\-a-c\end{bmatrix} \leftarrow \text{equal}$$

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)+T\left(\begin{bmatrix}c\\d\end{bmatrix}\right) = \begin{bmatrix}a-2b\\2a-3b\\-a\end{bmatrix} + \begin{bmatrix}c-2d\\2c-3d\\-c\end{bmatrix} = \begin{bmatrix}a+c-2b-2d\\2a+2c-3b-3d\\-a-c\end{bmatrix} \swarrow$$

T is comp $\overline{w}$ sum!

② $T\left(t\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}ta-2tb\\t2a-3tb\\-ta\end{bmatrix} \overset{?}{=} t \cdot T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) \nwarrow$

equal.

$t \cdot T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = t \cdot \begin{bmatrix}a-2b\\2a-3b\\-a\end{bmatrix} = \begin{bmatrix}ta-2tb\\2ta-3tb\\-ta\end{bmatrix} \leftarrow$ T is comp $\overline{w}$ scalar!

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b). Find matrix A that represents linear transformation :

$T: \mathbb{R}^2 \to \mathbb{R}^3$

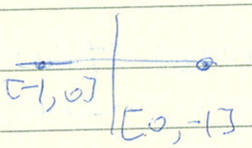
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-3y \\ 4x-5y \end{bmatrix} \qquad A = \begin{bmatrix} Te_1 & Te_2 \\ | & | \end{bmatrix}$$

$$Te_1 = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \qquad Te_2 = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \\ 4 & -5 \end{bmatrix}$$

c). Consider transformation B that consists of first rotating 180° in $\mathbb{R}^2$;
& applying Transformation. (T in b))

- Find the matrix representing B.
- The matrix that represents rotation of 180° in $\mathbb{R}^2$.



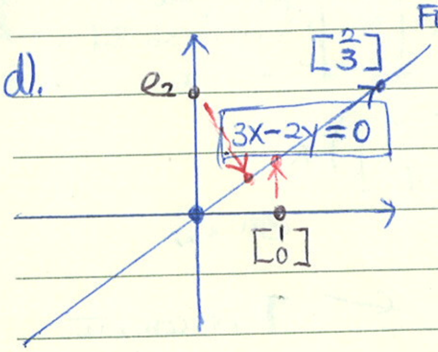
$$R = \begin{bmatrix} Re_1 & Re_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[rr, bend left, "B"] & \mathbb{R}^2 \arrow[r] & \mathbb{R}^3 \\ x \arrow[r] & Rx \arrow[r] & ARx \end{tikzcd} \qquad \boxed{B = AR}$$

$$B = \underset{A}{\begin{bmatrix} 1 & 2 \\ 1 & -3 \\ 4 & -5 \end{bmatrix}} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{Rot_\theta} = \begin{bmatrix} -1 & 2 \\ -1 & 3 \\ -4 & +5 \end{bmatrix}$$

Find matrix for projection on this ? $\frac{a \cdot b}{u^2}$ a

d).



$$\text{Proj}_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{2}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix}$$

$$P = \begin{bmatrix} 4/13 & 6/13 \\ 6/13 & 9/13 \end{bmatrix}$$

$$Pe_2 = \text{Proj}_{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{3}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix}$$

e). Find the matrix of linear. trans. that given a vector in $\mathbb{R}^2$ stretch it by factor of 7. Call matrix S.

$$S = \begin{bmatrix} Se_1 & Se_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

since $Se_1 = S\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}7\\0\end{bmatrix}$

$Se_2 = S\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\7\end{bmatrix}$

Example of least squares

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$Ax=b. \xrightarrow{\text{replace } \overline{w}} A^TAx = A^Tb$

- always has a solution
- if $Ax=b$ had solutions, $A^TAx=A^Tb$ has the same set of solutions.

$$\left.\begin{aligned} x+y&=1 \\ x+y&=2 \end{aligned}\right\} \text{Example}$$

$$\left.\underbrace{\begin{bmatrix}1&1\\1&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{b}\right\}$$

Normal Eq/ns :

$A^TAx = A^Tb.$

$$\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$$

$2x+2y=3$

$2x+2y=3$

$$\begin{bmatrix}2&2\\2&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$