**Cox rings and pseudoeffective cones on projectivized toric vector bundles**

Jose Gonzalez  -  University of Michigan

**Toric vector bundles**

A toric variety is a normal variety $X$ that contains a torus $T = \mathbb{G}_m^n$ as an open subset, together with an action of $T$ on $X$ that extends the natural action of $T$ on itself.\[ X = X(\Sigma), \] where $d = \dim X$, $n = \# \text{of rays in } \Sigma$, $k = \text{dim } T$.

A toric vector bundle is a vector bundle $F$ over a TVT with an action of $T$ such that the projection is equivariant and the action is linear on the fibers.

The main tool to study TVTs is the classification given by Klyachko:

**THEOREM** Klyachko’s category equivalence TVB

There are now various arguments showing that projectivized rank two TVBs are indeed finitely generated. The aim of this poster is to describe such an argument.

**Pic**($\Sigma$) on itself. \[ \text{Pic}(\Sigma) := \text{Cox}(\Sigma), \] and let $S = \{p_1, \ldots, p_n\}$ be the set of points corresponding to $F_1, \ldots, F_n$ in the projective space $\mathbb{P}(F_i)$.

**Theorem** The Cox ring $\text{Pic}(\Sigma)$ is isomorphic to a polynomial ring in $n$ variables over $\mathbb{R}$. The following result gives a negative answer to the finite generation question of Hering, Mustaţă and Payne.

**Theorem** Suppose that $\Sigma$ is uncountable and that $n > r > d$ and $\frac{d}{r} < \frac{1}{n}$. Then there is an irreducible toric vector bundle $F$ of rank $r$ on $X(\Sigma)$ such that the Cox ring of the projectivization $\mathbb{P}(F_i)$ is not finitely generated.

Moreover, even projectivized cotangent bundles of toric varieties are not Mori dream spaces in general.

**Pseudoeffective cones of projectivized TVBs**

[Gonzalez-Hering-Payne-Süß]

The pseudoeffective cones of projectivized TVBs are generated by torus invariant divisors, but in general they are not rational polyhedral.

**Lemma** Every effective divisor on $\mathbb{P}(F_i)$ is linearly equivalent to a torus invariant effective divisor.

**Theorem** Suppose that $\Sigma$ is uncountable, and that $n - d > r > d$ and $\frac{1}{r} < \frac{1}{n}$, and assume there is some cone $\sigma \in \Sigma$ such that every ray of $\Sigma$ is contained in either $\sigma$ or $-\sigma$. Then there is an irreducible toric vector bundle $F$ of rank $r$ on $X(\Sigma)$ such that the pseudoeffective cone of $\mathbb{P}(F_i)$ is not polyhedral.

**Notes:**

1. The variety in the Theorem: The columns of the following matrix span the rays of a smooth projective fan $\Sigma$ such that the projectivized cotangent bundle of the associated toric variety is not a toric dream space.

2. Note that the blow up of $\mathbb{P}^1$ at nine of these points is not a Mori dream space (Totaro). To get examples in higher dimensions, starting from the example given by a fan $\Sigma$ in dimension $d$, we embed $\mathbb{R}^d$ as the last coordinate hyperplane in $\mathbb{R}^{d+1}$, and let $S$ be the fan in $\mathbb{R}^d$ whose maximal cones are spanned by a maximal cone of $\Sigma$ together with either $(1, \ldots, 1, 0)$ or $(1, \ldots, 1, -1)$. The projectivized cotangent bundle of the associated toric variety is not a toric dream space.