Rouquier Lecture 3

Verma modules

$V \curvearrowleft W$, $\mathcal{A} = \{(c_s)\}$

$H \supset Z \supset P = \mathbb{C}[\mathcal{A} \times V/W \times V^*/W]$

$c \in \mathcal{A}$

$\overline{H}_c = H \otimes_P k_{c,0,0}$ – fin. dim. algebra, $\dim \overline{H}_c = |W|^3$

$= \dfrac{\mathbb{C}[V]}{(\mathbb{C}[V]^W_+)} \otimes \mathbb{C}W \otimes \dfrac{\mathbb{C}[V^*]}{(\mathbb{C}[V^*]^W_+)}$

$E \in \mathrm{Irr}\, W$, $\Delta(E) = H \otimes_{\mathbb{C}[V^*] \rtimes W} E$.

Baby Verma $\mathbb{C}_{c,0,0} \otimes_P \Delta(E) \in \overline{H}_c$-mod

$\downarrow$

$L(E)$ – unique irreducible

Thm (Gordon) $L(E) \leftarrow E$

$\overline{H}_c\text{-mod}_{gr} \xleftarrow{\sim} \mathrm{Irr}\, W$

$\overline{H}_c\text{-mod}_{gr}$ is hw category

Prop: Blocks are indexed by 2-sid. cells

$\mathcal{R} \supset \mathcal{R}^\circ_c$ – irred. comp of preimage – pt

$\downarrow$

Z

$\downarrow$

$\mathcal{P} \ni c \times 0 \times 0$

$\mathbb{C}(\mathcal{R}^\circ_c) \otimes_P H = \overline{H}$

blocks here are in bij-n w. 2-sid. cells

Γ – 2-sid. cells $\rightsquigarrow b_\Gamma$ – block

Def: c-families of Irr W = blocks of $L(E)$'s

Families for W Coxeter were defined by Lusztig (partly jt. w Kazhdan)

– same as blocks of HA of W over $\mathbb{Z}[q^{\pm 1}, \Phi_d(q)^{-1}]$ (cyclot. polyn)

Rouquier ≈ 1999 (b/c of isom. of this la-h w. Lusztig's J-algebra)

Hecke families ≠ CM-families (Thiel)

Decomp. matrices

Fix parameter c

	Irr Z		Irr H	
$\mathcal{A}\times V\times V^*$	$\mathcal{L}(w)$ ↓	$W \longleftrightarrow$	W	
$c\times V\times 0$	$\mathcal{L}(c)$ ↓	left cells ⟶		$\Delta(E)$ ↓
	$\mathcal{L}(\Gamma)$			
$c\times 0\times 0$	2-sided cells		Irr W	Baby Verma $\bar{\Delta}(E)$

irred comp ↓

$$\left[\mathbb{C}(\mathcal{R}^L_c)\otimes_{P}\Delta(E)\right]=\sum_{c \text{ left cell}} m_{E,c}\,[\tilde{\mathcal{L}}(c)] \quad \text{- def-n of } m_{E,c}$$

$$[\bar{\Delta}(E)]=\sum_{\substack{c\subset\Gamma\\ C}}\Big(\sum_{c\subset\Gamma} m_{Ec}\Big)[\tilde{\mathcal{L}}(\Gamma)], \qquad \tilde{\mathcal{L}}(c)=He\otimes_Z \mathcal{L}(c)$$

$E\in F(E)$ ⇓ ‖ ← happens to be

↑ family

$\dim E\cdot[\tilde{\mathcal{L}}(\Gamma)]$ - true in the ungraded cat-y.

Def: c - left cell; left cell rep. $\rho_c := \sum_{E\in \mathrm{Irr} W} m_{Ec}[E]$

Conj. W - Coxeter, then ρ_c is Lusztig's cell rep-n.
For $W=\mu_d^n\rtimes\mathfrak{S}_n$ we expect left cell reps related to level d to level d Fock space canonical basis. Here families are known

Prop $\bullet \sum_E m_{Ec}=|C|$; $\bullet \sum_c m_{Ec}=\dim E$ $\bullet$ $m_{EC}\neq 0 \Rightarrow E\in\mathcal{F}_\Gamma$

Prop: $[\mathrm{Soc}\ \mathrm{Res}_{H^-}\ \mathrm{Proj.cov}\ \tilde{\mathcal{L}}(c)]=\rho_c$

↑ annihilator of y's

$[b_c He/(S(V)^W_+) : {}^{b_c}\Delta(E)] \overset{?}{=} m_{EC}$

Proj. cover $\tilde{\mathcal{L}}(c)=b_c He/(S(V)^W_+)$ - filt. by $b_c\Delta(E)$

↑ tensor over $\mathbb{C}(\mathcal{R}^L_c)\otimes_P\cdot$

Prop. Given c $\exists!\, E \in \mathrm{Irr}\, W$ $m_{E_c} \neq 0$, s.t b_E is minimal where b_E is as follows:

$\mathbb{C}[V]/\mathbb{C}[V]^W_+$ - graded rep of $\mathbb{C}W$. b_E: min i s.t. E occurs in deg i

Prop: Γ - 2-sid cell $\exists!\, E \in \mathcal{F}_\Gamma$ (family) s.t. b_E is minimal (a.k.a. special rep of the family)

Smooth case: $\mathcal{Z}_c$ - CM space at c

$\mathcal{Z}' \times_{\mathcal{A}} \mathbb{C}_c$

Thm (E-G, Ginzburg-Kaledin, Gordon, Bellamy)

W-irred, $\exists\, c$ s.t $\mathcal{Z}_c$ smooth $\Leftrightarrow$ $W = \mu_d^n \rtimes S_n$ or $W = G_4$

Thm (EG, Gordon, BR)

$c \in \mathcal{A}$, Γ - 2-s. cell, $z_\Gamma \in \mathcal{Z}_c$ above $(c \times 0 \times 0)$

$\mathcal{Z}_c$ is smooth at $z_\Gamma \Leftrightarrow |\mathcal{F}_\Gamma| = 1$,

If this holds, then $b_\Gamma \overline{H}_c \sim e b_\Gamma \overline{H}_c e$ - schematic fiber of z_Γ

$|\Gamma| = \chi(1)^2$; $\chi \in \mathcal{F}_\Gamma$

$C \subset \Gamma$ - left cell, then $|C| = \chi(1)$, $\rho_C = \chi$

$\&(\mathbb{C}(\mathcal{R}_c^L) \otimes_P Z) b_c$ as a field ext-n: $(\mathbb{C}(c \times V \times 0) \otimes_P Z) b_c$

<u>Gaudin Hamiltonians</u>

$$\begin{array}{ccccc}
\mathcal{Z} & \hookleftarrow & \mathcal{Z}_{reg} & \longleftarrow & Y \\
\downarrow & & \downarrow & & \downarrow \psi \\
P & \hookleftarrow & P_{reg} & \longleftarrow & X \\
\| & & \| & & \| \\
\mathcal{A} \times V/W \times V^*/W & \supset & \mathcal{A} \times V_{reg}/W \times V^*/W & \longleftarrow & \mathcal{A} \times V_{reg} \times V^*/W
\end{array}$$

Prop: $Y = \{(c, v, u, \lambda) \in \mathcal{A} \times V_{reg} \times V^*/W \times V \mid$ ~~$\det(D_y - \langle \lambda, y \rangle)$~~ λ is e-value of D_y's $\forall y \in V$

$D_y \curvearrowright \mathbb{C}W$ dep on c, v, u, $[D_y, D_{y'}] = 0$

$D_y = \Theta^{-1}(y)$ - ~~[illegible]~~ Dunkl operator

$D_y: b_w \mapsto \langle y, w^{-1}(v)\rangle b_w + \sum_{s\in S} \varepsilon(s) c_s \frac{\langle y, \alpha_s\rangle}{\langle v, \alpha_s\rangle} b_{sw}$

Fix $c \in \mathfrak{h}$, $v\in V_{reg}$ (real case $v\in$ chamber), $v^*\in V^*_{reg}$ (real case: same chamber)

Starting pt $(0, v, v^*)$

Path $\gamma(t) = (tc, v, (1-t)v^*)$, $\gamma(0)=(0,v,v^*)$, $\gamma(1)=(c,v,0)$

Conj/Hypothesis: $\gamma(t)\in X_{\text{non-ramif for } y}$ $(t\neq 1)$ - conj for W Coxeter, c real

$\text{Spectrum}_{\gamma(t)}(D_y)$ is mult. free $t\neq 1$

$t=0$: $D_y: b_w \mapsto \langle y, w^{-1}(v^*)\rangle b_w$

$\text{Spec}_{\gamma(0)} \xrightarrow{\sim} W$

So $\text{Spec}_{\gamma(t)}(D_y) \xrightarrow{\sim} W \ \forall t\neq 1$

$\lambda_t(w) \longleftarrow w$

Prop: $w \sim_L w' \iff \lambda_1(w)=\lambda_1(w')$

Left cell $C \rightsquigarrow \lambda \in \text{Spec}_{\gamma(1)}(D_y)$

$\{a\in \mathbb{C}W \mid$ gen. e-vectors of this e-value$\} =$ ρ_c

when u is zero $D_y \curvearrowright \mathbb{C}W \curvearrowleft W$

eigenvalues at $\gamma(t) \xrightarrow[t\to 1]{}$ decomp-n of ρ_c into direct sum of lines

Q: rel-n to KL basis

Instead of $\mathbb{C}W$, there is filtered rep-n $L_{(c,m,u)} =$

$= \mathbb{C}_{c,m} \otimes H_{reg} e \otimes_{\mathbb{C}[V^*/W]} \mathbb{C}_u$; $\dim = |W|$

3.4 $c\in\mathfrak{h}$, $m\in V$, $u = V^*/W$

$\Theta^{-1}(\mathbb{C}[V^*]) \cap L_{(c,m,u)}$

at $c=0, u=0, m=0$: get $\mathbb{C}[V]/(\mathbb{C}[V]^W_+)$

$D'_y \curvearrowright gr\, L_{(c,m,u)}$ is as earlier

Fact: D'_y have same spectrum, for each left cell $\exists!$ single e-vector.

$\{D'_y\} \curvearrowright L$ - cyclic action

can be viewed as a vector in the comm. algebra $\xrightarrow{\sim}$ harmonic polyn

Q: • Rel-n of v_c w. [illegible] v_c

Goldie rk polynomial? or maybe basis of irred. comp-s?

• Do v_c give basis in simple rep-n?

$\bar{Y} \leftarrow Y$:

$\downarrow \quad \downarrow$ $\quad$ $\pi_1(\bar{X}(\mathbb{R})) \curvearrowright$ fibers

$\bar{X} \hookleftarrow X$ $\quad$ ↑ cactus group

↑ De Concini Procesi

A_n: Noah-White $\leadsto$ KL cells $\overset{?}{=}$ CM cell

Speculation c, Γ - 2-sid. cell

$z_\Gamma \in Z_c \quad \supset$ Slice Z_Γ

$\downarrow \qquad\qquad$ ↑

$c \times V/W \times V^*/W \qquad$ "type of Γ"

Q: how much of Lusztig theory of unip characters in the corresp family can be recovered from that?

? unip. characters, ? ~~Fourier~~ Fourier transform, ? Fusion cat-y