# Rouquier Lecture 1

I) Introduction

$G$-reduct. alg. gr-p ($G = GL_n$)

$\mathfrak{g} = Lie(G)$, $G(\mathbb{F}_q)$

$W$ = Weyl gr-p controls char. formulas, homol. properties of various cat-s of reps such as

- princ. block of cat. $\mathcal{O}$ for $\mathfrak{g}$ (KL polynomials, Soergel bimodules)
- Complex irred. reps of $G(\mathbb{F}_q)$
  $\cup$ unipotent irred. reps (Lusztig theory)

$\rightsquigarrow$ same results for $B_n, C_n$. (same Weyl group)

This goes via Hecke $\underbrace{\text{algebra}}_{\mathcal{H}(W)}$ (KL polynomials, etc)

$W \rightsquigarrow$ RCA. (for complex reflection groups)

Problem: Reconstruct categories of reps of $G(\mathbb{F}_q)$ (unip. part) / cat-y $\mathcal{O}$ for $\mathfrak{g}$ from $\tilde{\mathcal{H}}(W)$ (motivated by constructing for complex refl-n gr-ps the objects above) for $G(\mathbb{F}_q)$ Broue-Malle-Michel have extended Lusztig's combinatorial data) or Kazhdan-Lusztig theory ($\mathcal{H}(W)$ does exist but the question is whether KL basis makes sense)

Aim: ~~describe~~ conjectural descr-n of KL theory of cells (in Weyl groups)
cells - "limit" (crystal) of KL basis theory
- can be described via primitive ideals of $U(\mathfrak{g})$ (Joseph)

II) RCA: $V$ - fin. dim space / $\mathbb{C}$, $W \subset GL(V)$ - finite
$\cup$ $S$ - reflections (complex)

Assume: $W$ is gen-d by $S$. ($\Leftrightarrow S(V)^W$ is polyn ring)

Fix $\{c_s\}_{s\in S}$ - indeterminates $c_s = c_{s'}$, $s \underset{W}{\sim} s'$

$A = \mathbb{C}[\{c_s\}]$

Def (Etingof–Ginzburg) $\tilde{H} = A[t]\otimes T(V\oplus V^*)\rtimes W/\text{rel-ns}$

rel-ns: $[x,x']=[y,y']=0$, $x,x'\in V^*$, $y,y'\in V$

$[y,x] = t\langle y,x\rangle + \sum_{s\in S} c_s \langle s(y)-y, x\rangle s$

Thm (E.–G.) PBW decomp-n: have iso. of $A[t]$-vector spaces

$A[t]\otimes S(V)\otimes \mathbb{C}W\otimes S(V^*) \xrightarrow[\text{mult}]{\sim} \tilde{H}$

i.e. $\tilde{H}$ is a flat family of algebras / Spec $A[t]$

$c=0$, $t=0$: $S(V\oplus V^*)\rtimes W$

For $t=1$: $\tilde{H}/(t-1) =: H'$

Given: $c: A\to\mathbb{C} \rightsquigarrow H'_c$

$\mathcal{O}_c = \{$fin. gen-d $H'_c$-modules w. loc. nilp. action of $V\}$

Thm (Ginzburg–Guay–Opdam–Rouquier) $\mathcal{O}_c$ is a highest weight cat-y

$\mathrm{Irr}(\mathcal{O}_c) \xrightarrow{\sim} \mathrm{Irr}(W)$

$H'$ is deformation of $H'_0 = \mathcal{D}(V)\rtimes W$

(-nontriv. deformm-n thanks to presence of $W$)

$V\to V/W$ ramified, $\alpha_s\in V^*$, $\ker\alpha_s = \ker(s-1)$

$\delta = \prod \alpha_s \in S(V^*)$

$V_{reg} = V\setminus(\delta=0)$ - complement of the branch locus

$\mathbb{C}[V_{reg}] = \mathbb{C}[V][\delta^{-1}]$

$H'_{reg} = H'[\delta^{-1}]$

Fact: $H'_{reg} \simeq A\otimes\mathcal{D}(V^{reg})\rtimes W$

$y\in V \rightsquigarrow D_y = \frac{\partial}{\partial y} - \sum_{s\in S} \underbrace{\varepsilon(s)}_{\text{e-value}} c_s \langle y,\alpha_s\rangle \alpha_s^{-1} s \in A\otimes\mathcal{D}(V^{reg})\rtimes W$

Thm (E.–G.) Isom. of $A$-alg-s $H'_{reg}\xrightarrow{\sim} A\otimes\mathcal{D}(V_{reg})\rtimes W$

$y\mapsto D_y$ and is identity on $A\otimes\mathbb{C}[V_{reg}]\rtimes W$

Can be used to construct KZ functor    b/c $H_{\text{"}\exp(c)\text{"}}(W) = \mathbb{C}B_W/\text{rel-ns}$

$$\begin{tikzcd}
KZ: \mathcal{O}_c \arrow[r] & H_{\text{"}\exp(c)\text{"}}(W)\text{-mod} \arrow[rd, bend left] & \\
H'_c\text{-mod} \arrow[u, phantom, "\cap" rotate=90] \arrow[r] \arrow[d] & \text{flat vector bundle} \arrow[r, "\text{De Rham}"'] & \pi_1(V_{reg}/W)\text{-mod} \\
H'_{c,reg}\text{-mod} = \mathcal{D}\text{-mod}_{V^{reg}/W} \arrow[ru, phantom, "\cap"] & &
\end{tikzcd}$$

Thm (GGOR) KZ is fully faithful on proj-s

so $\mathcal{O}_c\text{-proj} \hookrightarrow H_{\text{"}\exp(c)\text{"}}(W)\text{-mod}$

Dec. numbers (~~#~~ mult-s of simples in standards) known for $W = \mu_d \wr \mathfrak{S}_n$

Digression: classif-n of irred. complex refl-n groups

$G(d,e,n)$ $n \geq 1$, $d \geq 1$, $e \mid d$

+34 exceptional groups

$G(d,1,n)$ = monomial $n \times n$ matrices w. coeff-s in $\mu_d$ (= $d$th roots of 1)

$G(d,e,n) = \{$el-ts in $G(d,1,n) \mid$ product of nonzero entries is root of 1 of power $d/e\} \underset{\substack{\uparrow \\ \text{index } e}}{\subseteq} G(d,1,n)$

$W = \mathfrak{S}_n \Rightarrow \mathcal{O}_c \simeq q\text{-Schur}_n\text{-mod}$

$q = \exp(c)$

There's a mod $\ell$-version of $q\text{-Schur}_n \subseteq \overline{\mathbb{F}}_\ell GL_n(\mathbb{F}_q)\text{-mod}_{unip}$

(Dipper-James)

Takeuchi: image ~~of~~ over $\mathbb{Z}_\ell$:

$$\mathbb{Z}_\ell GL_n(\mathbb{F}_q) \Big/ \Big( \bigoplus_{\chi\text{-noncusp}} e_\chi \mathbb{Q}_\ell GL_n(q) \Big) \cap \mathbb{Z}_\ell GL_n(q)\text{-mod}$$

• $t=0$ : $H = \tilde{H} \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t)$

$c \rightsquigarrow H_c$

$H_0 = \mathbb{C}[V \oplus V^*] \rtimes W$, $Z(H_0) = \mathbb{C}[V\oplus V^*]^{\Delta W}$ ($\Delta : W \to W \times W$ diagonal)

$Z = Z(H)$

<u>Thm</u> (EG) • Satake iso: $Z \xrightarrow{\sim} \underset{\mathrm{End}_H(He)}{\overset{}{eHe}}$ , $e = \frac{1}{|W|}\sum_{w\in W} w$.

$$\cdot\ H \xrightarrow{\sim} \mathrm{End}_Z(He)^{op}$$

<u>Thm</u> Have iso of $A$-algebras

$$\underset{H\otimes_{\mathbb{C}[V]}\mathbb{C}[V_{reg}]}{\overset{\shortparallel}{H_{reg}}} \xrightarrow{\sim} A \otimes \mathbb{C}[V_{reg} \times V^*] \rtimes W$$

(~~remove $\frac{\partial}{\partial y}$~~ in the $t=1$ isomorphism, replace $\frac{\partial}{\partial y}$ w. $y$)

$$P = A \otimes \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W \subset H$$
$$\subset Z \subset$$

$H$ is free $P$-module of rk $|W|^3$

$Z$ is free $P$-module of rk $|W|$

$R$- commut. $P$-alg $\Rightarrow Z(H\otimes_P R) = Z \otimes_P R$ (b/c $Z$ is a free summand of $H$ as an $Z$-module)

Exercise $H = Z \oplus ((1-e)H + H(1-e))$

Thm: if $Z\otimes_P R$ is regular, then $H \otimes_P R$ is Morita equivalent to $Z \otimes_P R$.

<u>Example</u> $R = P \otimes_{\mathbb{C}[V]^W} \mathbb{C}[V^{reg}]^W$, have $Z_{reg} = A \otimes \mathbb{C}[V_{reg} \times V^*]^{\Delta W}$

Question: Study rep. theory of $H$ as $P$-algebra

$K = \mathrm{Frac}(P)$

$\rightsquigarrow K \otimes_P H \underset{\text{Morita}}{\sim} K \otimes_P Z = \mathrm{Frac}(Z) =: L$

$\Rightarrow K \otimes_P H = \mathrm{Mat}_{|W|}(L)$

$K\otimes_P H$ has a unique simple module, dim is $|W|$
it is NOT absolutely irreducible
$M :=$ Galois closure of $L/K$
Consider reps of $M\otimes_P H$ = product of matrix alg-s$/M$
We'll see $\mathrm{Irr}(M\otimes_P H) \xrightarrow{\sim} W$ (lecture 2)

Or can take the point $(0,0)\in V/W\times V^*/W$.

$$\rightsquigarrow (0,0,c)\in \underbrace{V/W\times V^*/W}_{=\,\mathrm{Spec}(A)}\times \mathfrak{A} := \underbrace{\mathcal{P}}_{=\,\mathrm{Spec}(P)}$$

$\mathfrak{m}$ - max. ideal
Gordon: $\mathrm{Irr}\,(H/\mathfrak{m}H) \xrightarrow{\sim} \mathrm{Irr}\, W$.

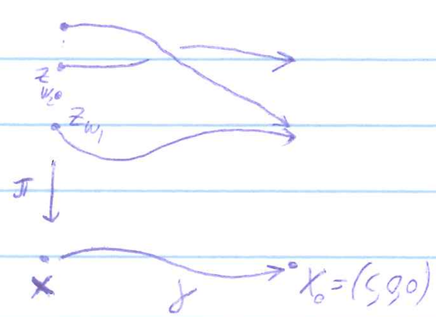
Decomp. matrices $\rightsquigarrow$ cells, left cell reps, families

Topological picture:

$$\begin{array}{ccl} (V\times V^*)/W \subset & \mathcal{Z} = \mathrm{Spec}(Z) & (\text{Calogero-Moser}) \\ \downarrow & \pi\downarrow & \text{finite flat map} - \cancel{\text{gen}}\ \text{degree} = |W| \\ V/W\times V^*/W \subset & \mathcal{P} = \mathfrak{A}\times V/W\times V^*/W & \end{array}$$

$c$-parameter $x = (c, x_1, x_2)$
$z\in\mathcal{Z}$, $\pi(z) = (c, x_1, x_2)$ s.t. $\pi^{-1}(x) \xrightarrow{\sim} W$

$$z_w \longleftarrow\!\!\shortmid\ w \in W$$ - needs to be done carefully!




path staying in the non-ramified part $\rightsquigarrow$ can be lifted uniquely

$\overline{Z}_W$ - end-points

two-sided cells $\xleftrightarrow{\sim}$ pts in $\pi^{-1}(\varsigma, 0, 0)$

$\rightsquigarrow$ two-sided cells

**Left cells**: $x_L = (\varsigma, v, 0)$, $v \in V_{reg}/W$.

$x_R = (\varsigma, 0, v^*)$

$$\begin{array}{c} \mathcal{Z}_{reg} = Z \times_{V/W} V^{reg}/W \xrightarrow{\sim} A \times \cancel{V_{reg}/W \times V^*} (V_{reg} \times V^*)/W \\ \downarrow \pi_{reg} \\ A \times V_{reg}/W \times V^*/W \end{array}$$

Exer: $W = \mathbb{Z}/3\mathbb{Z}$