MATH 603, PROBLEM SET 3, DUE APR 7

IVAN LOSEV

There are five problems worth 25 points total. You need to score 15 points to get the maximal score. You can use previous problems (or previous parts) in your solutions of the subsequent problems (or subsequent parts of the same problem) and get full credit even if you haven't solved the problems/parts you have used. Partial credit is given. The italized text serves as comments to a problem, but it is not a part of the problem.

The solutions need to be submitted via Canvas. Hand-written solutions are accepted but please make sure they are readable.

Here's a comment on the status of the problems below. They all deal with the so called "category $\mathcal{O}$"– to be formally introduced later. The first is about the "singular block". This is the same phenomenon as in problems 3b and parts 4,5 of problem 4 of HW2. The second problem deals with the tensor products of Verma modules with finite dimensional representations – yes, tensor products are important! The third problem deals with the duality functor, some parts are similar to Problem 5 in HW2 – and this is not a coincidence. The fourth problem is about a remarkable object, known as "the anti-dominant projective" or "the big tilting".

These problems are important for the general theory and will be referred to in our treatment of the category $\mathcal{O}$ in the third, "Hecke", part of the course. They are given as homework problems partly due to lack of time. The last problem will not be used – but our old friend, the degenerate affine Hecke algebra, makes the appearance there.

Some notation and terminology. Below $\mathfrak{g}$ denotes $\mathfrak{sl}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed characteristic 0 field, and $\mathfrak{h}$ is the Cartan subalgebra. By a weight $\mathfrak{g}$-module we mean a $\mathfrak{g}$-representation M with $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where $\mathfrak{h}$ acts on M_λ by λ and $\dim M_\lambda < \infty$.

Let $\lambda' \in \Lambda$ and χ denote the orbit $W \cdot \lambda'$ so that for every element $z \in Z$, the center of $U(\mathfrak{g})$, the value $\mathrm{HC}_z(\chi) := \mathrm{HC}_z(\lambda)$ is well-defined. We write M^χ for the submodule

$$\{m \in M | \forall z \in Z | \exists k > 0 \text{ s.t. } (z - \mathrm{HC}_z(\chi))^k m = 0\}.$$

Then, M^χ is a weight module and $M = \bigoplus_\chi M^\chi$ (the reason: z preserves all M_λ, and we can decompose them into the direct sum of generalized eigenspaces because they are finite dimensional.

Problem 1, 4pts total. 1, 2pts) Show that $\Delta(-\rho)$ is irreducible.

2, 2pts) Now suppose that M has a filtration $\{0\} \subset M_1 \subset M_2 \subset \ldots \subset M_k = M$ such that $M_i/M_{i-1} \cong \Delta(-\rho)$ for all i. Show that M is completely reducible.

Problem 2, 6pts total. *The goal of this problem is to investigate the modules of the form $V \otimes \Delta(\lambda)$ for $\lambda \in \Lambda$, where V is a finite dimensional representation of $\mathfrak{g}$. We will elaborate on their importance when we study the category $\mathcal{O}$ and the Soergel theory. For now, the right way to think about what we are doing is that we are producing some (very interesting and useful) infinite dimensional representations of $\mathfrak{g}$.*

1, 1pt) Show that $V \otimes \Delta(\lambda)$ is a weight $\mathfrak{g}$-module in the terminology above.

2, 1pt) Show that the functor $V\otimes\bullet : U(\mathfrak{g})\text{-mod} \to U(\mathfrak{g})\text{-mod}$ is left adjoint to $V^*\otimes\bullet$ (*for students who have, hmm..., happy memories of MATH 380 – you need to establish a natural isomorphism, but you don't need to check the fun and exciting commutative diagrams*).

Our next goal is to establish a filtration on $V\otimes\Delta(\lambda)$ whose successive quotients are Verma modules. We write $\mathfrak{b}$ for the subalgebra of $\mathfrak{g}$ consisting of all upper triangular matrices. Note that we have a natural projection $\mathfrak{b}\to\mathfrak{h}$. For $\lambda\in\Lambda$, let $\mathbb{F}_\lambda$ denote the one-dimensional representation of $\mathfrak{b}$ obtained as $\mathfrak{b}\to\mathfrak{h}\xrightarrow{\lambda}\mathbb{F}$. Consider the induction functor $\Delta : U(\mathfrak{b})\text{-mod}\to U(\mathfrak{g})\text{-mod}, U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\bullet$. Observe – you don't need to prove this – that $\Delta(\lambda)=\Delta(\mathbb{F}_\lambda)$.

3, 1pt) Note that one can view V as a $\mathfrak{b}$-module. Establish a functor isomorphism $V\otimes\Delta(\bullet)\cong\Delta(V\otimes\bullet)$ (*Hint: Yoneda! And no need to check commutative diagrams*).

4, 1pt) Prove that $U(\mathfrak{g})$ is a free right $U(\mathfrak{b})$-module (*hint: PBW!*) and deduce that Δ is an exact functor.

5, 1pt) *Now we are going to establish a filtration on $V\otimes\Delta(\lambda)$ whose successive quotients are Verma modules.* Pick a weight basis $v_1,\ldots,v_m$ of V of weights $\mu_1,\ldots,\mu_m$. We choose an ordering so that $\mu_i\leqslant\mu_j$ implies $i\geqslant j$ (*somewhat informally, the order is decreasing*). Prove that there is a filtration $\{0\}=M_0\subset M_1\subset\ldots\subset M_m=V\otimes\Delta(\lambda)$ such that $M_i/M_{i-1}\cong\Delta(\lambda+\mu_i)$ (*hint: Produce a filtration by $\mathfrak{b}$-submodules on V – Section 3 of Lecture 11 may provide some inspiration – and realize M_i as the images of filtration terms there under Δ*).

6, 1pt) *Now we study the modules $(V\otimes\Delta(\lambda))^\chi$.* In the notation of the previous part, show that M_i^χ/M_{i-1}^χ is $\Delta(\lambda+\mu_i)$ if $\lambda+\mu_i\in\chi$ and is zero else.

Problem 3, 7pts total. *This problem deals with the duality for the weight modules. What we are going to produce should be thought as an analog of the duality for finite dimensional vector spaces.* First, define the <u>restricted dual</u> $M^\vee := \bigoplus_{\lambda\in\Lambda} M_\lambda^*$.

1, 1pt) Show that $M^\vee$ is a $\mathfrak{g}$-subrepresentation of M^* and $(M^\vee)_\lambda = M^*_{-\lambda}$.

Note that $M\mapsto M^\vee$ is a contravariant functor from the category of weight $\mathfrak{g}$-modules to itself. Observe that $x\mapsto -x^t$ (transpose) is an automorphism of $\mathfrak{g}$. We define $\mathbb{D}M$ as $M^\vee$ with the action of $\mathfrak{g}$ twisted by the automorphism $x\mapsto -x^t$.

2, 1pt) *The twist by this automorphism is justified by the following observation.* Identify $(\mathbb{D}M)_\lambda$ with M_λ^*.

3, 1pt) Establish a functor isomorphism $\mathrm{id}\xrightarrow{\sim}\mathbb{D}^2$ (both functors are endo-functors of the category of weight modules).

4, 1pt) Prove that $\mathbb{D}L(\lambda)\cong L(\lambda)$ for all $\lambda\in\Lambda$.

5, 1pt) For a finite dimensional representation V, establish a functor isomorphism $\mathbb{D}(V\otimes\bullet)\cong V\otimes\mathbb{D}(\bullet)$.

6, 1pt) Establish a functor isomorphism $\mathbb{D}(\bullet^\chi)\cong(\mathbb{D}\bullet)^\chi$.

7, 1pt) For $\mu\in\Lambda$, define the *dual Verma module*

$$\nabla(\mu) := \mathbb{D}\Delta(\mu).$$

Prove that $\dim\operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda),\nabla(\mu))=\delta_{\lambda,\mu}$ and that every exact sequence of the form

$$0\to\nabla(\mu)\to V\to\Delta(\lambda)\to 0$$

splits. *Note that this is a complete analog of the last two parts in Problem 5 of PSet 2.*

Problem 4, 4pts total. Let $\chi := W \cdot 0$. We consider the module $M := (L(\rho) \otimes \Delta(-\rho))^{\chi}$.

1, 1pt) Show that M admits a filtration, where the successive quotients are all $\Delta(w \cdot 0)$ with $w \in W$, each occurring once, and moreover, $\Delta(-2\rho)$ arises as the top quotient, i.e., the quotient of M by the next filtration term (*note that* $-2\rho = w_0 \cdot 0$, *where* $w_0 \in S_n$ *is defined by* $w_0(i) = n + 1 - i$).

2, 1pt) Show that for $\mathfrak{sl}_2$, the module of interest is $\mathbb{C}^2 \otimes \Delta(-1)$. Describe the subobject $\Delta(0) \subset M$ and an isomorphism $M/\Delta(0) \xrightarrow{\sim} \Delta(-2)$ explicitly (*e.g. you need to realize* $\Delta(0)$ *as a subspace*).

3, 1pt) *This is harder.* Show that $\dim \mathrm{End}_{\mathfrak{g}}(M) = |W|$ (*hint: you should use multiple parts of problem 3*).

4, 1pt) Show that for $\mathfrak{g} = \mathfrak{sl}_2$, this endomorphism algebra is isomorphic to $\mathbb{F}[x]/(x^2)$, where x arises from the action of the Casimir.

Problem 5, 4pts. Let M be a representation of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ (w/o any conditions). Let $V := \mathbb{F}^n$ be the tautological representation. The goal of this problem is to establish an action of the degenerate affine Hecke algebra $\mathcal{H}(d)$ on $M \otimes V^{\otimes d}$ by automorphisms. Consider the element $\tilde{C} := \sum_{i,j=1}^n E_{ij} \otimes E_{ji} \in \mathfrak{g} \otimes \mathfrak{g}$.

1, 1pt) Show that $\tilde{C}$ is annihilated by the action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$. Deduce that for any $\mathfrak{g}$-representations V_1, V_2 the operator $V_1 \otimes V_2 \to V_1 \otimes V_2$ given by $\tilde{C}$, i.e., $v_1 \otimes v_2 \mapsto \sum_{i,j=1}^n E_{ij} v_1 \otimes E_{ji} v_2$ is $\mathfrak{g}$-linear.

2, 1pt) Show that $\tilde{C}$ acts on $V \otimes V$ by the permutation of tensor factors, $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$.

3, 2pts) Now consider the following operators on $M \otimes V^{\otimes d}$. First, we have the s_i for $i = 1, \ldots, n-1$, permuting the tensor factors i and $i+1$ in $M \otimes V^{\otimes d}$. Then, for $j = 1, \ldots, n$, we have X_j defined as follows. We can write $M \otimes V^{\otimes d}$ as $(M \otimes V^{\otimes (j-1)}) \otimes V \otimes V^{\otimes d-j}$. Then X_j is obtained by applying $\tilde{C}$ to the first and second factors, explicitly, $X_j(u_1 \otimes u_2 \otimes u_3) = \sum_{i,j=1}^n E_{ij} u_1 \otimes E_{ji} u_2 \otimes u_3$ for $u_1 \in M \otimes V^{\otimes (j-1)}, u_2 \in V, u_3 \in V^{\otimes d-j}$. Show that the operators $X_1, \ldots, X_d, s_1, \ldots, s_{d-1}$ define a representation of the degenerate affine Hecke algebra $\mathcal{H}(d)$ in $M \otimes V^{\otimes d}$ by $\mathfrak{g}$-linear endomorphisms.

Aside. We'll see later that the objects from Problems 1-4 play an important role in the Soergel theory. Here's one brief glance. It turns out that the endomorphism algebra of M in Problem 4 is $\mathbb{F}[x_1, \ldots, x_n]/(\mathfrak{m}_0)$, where $\mathfrak{m}_0$ is the maximal ideal in the algebra of symmetric polynomials $\mathbb{F}[x_1, \ldots, x_n]_{sym}$ of all polynomials without constant term. By the aside to PSet 1, this quotient indeed has dimension $|W| = n!$. The homomorphism $Z \to \mathrm{End}_{\mathfrak{g}}(M)$ is surjective (part 4) of Problem 4 is a special case).

By the way, here is another appearance of $\mathbb{F}[x_1, \ldots, x_n]/(\mathfrak{m}_0)$ – it is the cohomology algebra of the flag variety Fl_n of complete flags of subspaces in $\mathbb{C}^n$. And this is very much relevant to what we study, although to explain the geometric side of this picture will have to wait until the very end of the class...