REPRESENTATION THEORY, CHAPTER 0. BASICS

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1. Generalities on associative algebras and their modules

1.1. **Basic definitions.** In this section we define associative algebras, their left and right modules and bimodules. All rings are associative and unital. Let $\mathbf{k}$ be a commutative ring. We will mostly be interested in the case when $\mathbf{k}$ is an algebraically closed field.

Definition 1.1. By an (associative, unital) $\mathbf{k}$-algebra we mean an (associative, unital) ring A together with a ring homomorphism $\iota : \mathbf{k} \to A$ such that $\iota(r)a = a\iota(r)$ for all $r \in \mathbf{k}, a \in A$.

In particular, any ring is a $\mathbb{Z}$-algebra.

Definition 1.2. A homomorphism of $\mathbf{k}$-algebra is a $\mathbf{k}$-linear homomorphism of unital rings.

Definition 1.3. Let A, B be (associative, unital) rings and M be an abelian group.

- By a (left) A-module structure on M we mean a $\mathbb{Z}$-bilinear *multiplication* map $A \times M \to M$ that is associative in the sense that $a_1(a_2m) = (a_1a_2)m$ for all $a_1, a_2 \in A, m \in M$.

- By a right B-module structure on M we mean a $\mathbb{Z}$-bilinear *multiplication* map $M \times B \to M$ that is associative in the sense that $(mb_1)b_2 = m(b_1b_2)$.
- By an A-B-bimodule structure on M we mean a pair of a left A-module and a right B-module structures such that $a(mb) = (am)b$.

Remark 1.4. Let A^{opp} denote the ring that is the same as A as an abelian group, but the multiplication is in the opposite order. Then a right A-module is the same thing as a left A^{opp}-module (and vice versa).

Definition 1.5. Let M, N be left A-modules. By an A-module *homomorphism* (a.k.a. A-linear map) $\varphi : M \to N$ one means an abelian group homomorphism $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A, m \in M$.

1.2. **Basic examples and constructions of modules.** In this section $\mathbf{k}$ is a commutative ring and A is an $\mathbf{k}$-algebra.

Example 1.6. A can be viewed as a left A-module, as a right A-module and as an A-A-bimodule via the multiplication. These (bi)modules are called *regular*.

Definition 1.7. Let M_1, M_2 be two left modules. We can define their direct sum $M_1 \oplus M_2$ in the standard way: it consists of pairs (m_1, m_2), where $m_i \in M_i$, with componentwise operations. More generally, for an index set $\mathcal{I}$ (which may be finite or infinite) and left A-modules $M_i, i \in \mathcal{I}$ we define their direct sum $\bigoplus_{i \in \mathcal{I}} M_i$ as the set consisting of all collections $(m_i)_{i \in \mathcal{I}}$ with only finitely many nonzero entries (with componentwise operations).

Example 1.8. Let $\mathcal{I}$ be an index set. We can form the *coordinate A-module* $A^{\oplus I}$.

Now we proceed to submodules and quotient modules.

Definition 1.9. Let M be a (left) A-module. By a *submodule* of M we mean a nonempty subset closed under the module operations: addition and the multiplication by elements of A. One can define submodules in right modules and subbimodules in bimodules in a similar fashion.

Example 1.10. The submodules of the regular left A-module A are exactly the left ideals of A. For the regular right A-module (resp., A-bimodule), we arrive at right (resp., two-sided) ideals.

Let M be a left A-module and M_0 be its submodule. We can form the quotient abelian group M/M_0 and endow it with the unique module structure with the multiplication by elements of A given by $a(m + M_0) := am + M_0$. The resulting module is called the *quotient* (of M by M_0). The following lemma states the universal property of a quotient module. Note that we have a natural surjective map $M \to M/M_0, m \mapsto m + M_0$, denote it by π.

Lemma 1.11. *Let M, N be A-modules and $M_0 \subset M$ be a submodule. Let $\varphi : M \to N$ be an A-linear map sending M_0 to zero. Then there is a unique A-linear map $\underline{\varphi} : M/M_0 \to N$ such that $\varphi = \underline{\varphi} \circ \pi$. It is given by $\underline{\varphi}(m + M_0) := \varphi(m)$.*

As for vector spaces, it makes sense to speak about spanning sets and bases of modules. Clearly, every module has a spanning set (we can take all elements of the module, for example). Unlike for vector spaces, not every module has a basis. A module that has a basis is called *free*.

Example 1.12. The coordinate module $A^{\oplus \mathcal{I}}$ is free. It has the *coordinate* basis: $e_i := (\delta_{ij})_{j \in \mathcal{I}}$, where δ_{ij} is the Kroneker delta.

The following proposition describes free modules and shows that every module is a quotient of a free module.

Proposition 1.13. *Let M be a left A-module, $\mathcal{I}$ be an index set, and $m_i \in M, i \in \mathcal{I}$, a collection of elements. The following claims are true:*

1. *There is a unique A-linear map $A^{\oplus \mathcal{I}} \to M$ such that $e_i \mapsto m_i$.*
2. *This map is surjective if and only if the elements m_i span M. In particular, every M is isomorphic to a quotient of a free module.*
3. *This map is an isomorphism if and only if the elements m_i form a basis of M. In particular, every free module is isomorphic to a coordinate module.*

We note that this proposition holds for right modules, but needs to modified for bimodules.

Example 1.14. Suppose M is spanned by a single element m. Then M is isomorphic to A/I, where I is the left ideal $\{a \in A | am = 0\}$.

1.3. **Basic examples and constructions of algebras.** We start by giving some examples of $\mathbf{k}$-algebras, where $\mathbf{k}$ is a commutative (as well as associative and unital) ring.

Example 1.15. Let n be a non-negative integer and A be a $\mathbf{k}$-algebra. We can form the $\mathbf{k}$-algebra $\mathrm{Mat}_n(A)$ of $n \times n$-matrices with the usual matrix multiplication.

Example 1.16. Let G be a group. Then we can form its *group algebra* $\mathbf{k}G$ defined as follows. It is a free module with basis identified with the elements of G (so that every element of $\mathbf{k}G$ is uniquely written as $\sum_{g \in G} a_g g$ for $a_g \in \mathbf{k}$). The multiplication of the basis elements is the same as in the group, it is extended to $\mathbf{k}G$ by bilinearity.

The importance of this construction is as follows. For a group G and an (associative and unital) algebra B consider the set of maps $G \to B$ that sends 1 to 1 and respect the multiplication. Then this set is in a natural bijection with the set of $\mathbf{k}$-algebra homomorphisms $\mathbf{k}G \to B$. The map from the former to the latter is the extension by linearity and the map in the opposite direction is the restriction to G.

In particular, let V be a $\mathbf{k}$-vector space and $B := \mathrm{End}(U)$. A map $G \to \mathrm{End}(U)$ as above is nothing else as a representation of G in U, while a $\mathbf{k}$-algebra homomorphism $\mathbf{k}G \to \mathrm{End}(U)$ is a $\mathbf{k}G$-module structure on U. So, a representation of G (over $\mathbf{k}$) is the same thing as a $\mathbf{k}G$-module.

Example 1.17. Let A be a $\mathbf{k}$-algebra and let I be a two-sided ideal. Then the quotient A/I has a natural algebra structure. It satisfies a universal property similar to Lemma 1.11 (but for homomorphisms of algebras).

Example 1.18. Let A_1, A_2 be two $\mathbf{k}$-algebras. Then their direct sum $A_1 \oplus A_2$ has a natural algebra structure. The same is true for the direct sum of finitely many $\mathbf{k}$-algebras. However, the direct sum of infinitely many algebras cannot be unital.

1.4. **Module of homomorphisms.** Let $\mathbf{k}$ be a commutative ring, A be a $\mathbf{k}$-algebra and M, N be left $\mathbf{k}$-modules. Let $\mathrm{Hom}_A(M, N)$ denote the set of A-module homomorphisms $M \to N$. This set carries a natural $\mathbf{k}$-module structure via:

$$[\varphi_1 + \varphi_2](m) := \varphi_1(m) + \varphi_2(m);$$
$$[r\varphi_1](m) := r(\varphi(m)), \quad r \in \mathbf{k}, \varphi_1, \varphi_2 \in \mathrm{Hom}_A(M, N), m \in M.$$

Exercise 1.19. Check that $\mathrm{Hom}_A(M, N)$ is indeed a $\mathbf{k}$-module.

For three left A-modules L, M, N we have the composition map

$$\mathrm{Hom}_A(M,N)\times \mathrm{Hom}_A(L,M)\to \mathrm{Hom}_A(L,N), (\varphi,\psi)\mapsto \varphi\circ\psi.$$

It is $\mathbf{k}$-bilinear.

In particular, let N_2 be an A-module, $N_1\subset N_2$ be an A-submodule, and $N_3 := N_2/N_1$ be the quotient module. Let $\iota : N_1 \hookrightarrow N_2$ denote the inclusion map, and $\pi : N_2 \twoheadrightarrow N_3$ denote the projection map. This gives rise to the $\mathbf{k}$-linear maps

$$\tilde{\iota} : \mathrm{Hom}_A(M,N_1)\to \mathrm{Hom}_A(M,N_2), \varphi_1\mapsto \iota\circ\varphi_1,$$
$$\tilde{\pi} : \mathrm{Hom}_A(M,N_2)\to \mathrm{Hom}_A(M,N_3), \varphi_2\mapsto \pi\circ\varphi_2.$$

The following lemma is important in understanding the behavior of the Hom modules.

Lemma 1.20. *The map $\tilde{\iota}$ is injective, while* $\operatorname{im}\tilde{\iota} = \ker\tilde{\pi}$.

Proof. The claim that $\tilde{\iota}$ is injective is left as an exercise. Note that $\pi\circ\iota = 0$, hence $\tilde{\pi}\circ\tilde{\iota}(\varphi_1) = \pi\circ\iota\circ\varphi_1 = 0$. So $\tilde{\pi}\circ\tilde{\iota} = 0$, equivalently, $\operatorname{im}\tilde{\iota}\subset\ker\tilde{\pi}$. Conversely, take $\varphi_2\in\ker\tilde{\pi}$, equivalently, such that $\pi\circ\varphi_2 = 0$. So $\operatorname{im}\varphi_2\subset\ker\pi = N_1$. Let φ_1 denote φ_2 viewed as an A-linear map $M\to N_1$. By the construction, $\varphi_2 = \iota\circ\varphi_1 = \tilde{\iota}(\varphi_1)$. We conclude that $\ker\tilde{\pi}\subset\operatorname{im}\tilde{\iota}$. This finishes the proof. □

Now suppose that B is another $\mathbf{k}$-algebra and M be an A-B-bimodule. Then, for any A-module M, the $\mathbf{k}$-module $\mathrm{Hom}_A(M,N)$ upgrades to a left B-module via:

$$[b\varphi](m) := \varphi(mb).$$

Similarly, if C is a $\mathbf{k}$-algebra and N is an A-C-bimodule, then, for any left A-module M, the $\mathbf{k}$-module $\mathrm{Hom}_A(M,N)$ upgrades to a right C-module:

$$[\varphi c](m) := \varphi(cm).$$

And if M is an A-B-bimodule, and N is an A-C-bimodule, then $\mathrm{Hom}_A(M,N)$ is a B-C-bimodule.

Now let M be an A-module. We write $\mathrm{End}_A(M)$ for $\mathrm{Hom}_A(M,M)$. The composition endows $\mathrm{End}_A(M)$ with the structure of an associative $\mathbf{k}$-algebra. The identity map is a unit. In particular, for $M = A^{\oplus n}$, we have $\mathrm{End}_A(M) = \mathrm{Mat}_n(A^{opp})$.

In the general case, M becomes an A-$\mathrm{End}_A(M)^{opp}$-bimodule. In particular, for two left A-modules M, N, the $\mathbf{k}$-module $\mathrm{Hom}_A(M,N)$ upgrades to an $\mathrm{End}_A(N)$-$\mathrm{End}_A(M)$-bimodule, where the actions are by taking compositions.

1.5. **Tensor product of modules and algebras.** Let $\mathbf{k}$ be a commutative ring and A be a $\mathbf{k}$-algebra. In the previous section from two left A-modules M, N we have produced a $\mathbf{k}$-module $\mathrm{Hom}_A(M,N)$. Same works when both M and N are right A-modules.

Now let M be a right A-module and N be a left A-module. We will produce an $\mathbf{k}$-module $M\otimes_A N$, the tensor product of M and N.

First, we explain the universal property $M\otimes_A N$ is supposed to satisfy. For this we need the notion of a bilinear map in this setting.

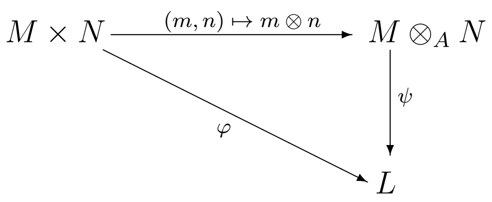
Definition 1.21. Let L be a $\mathbf{k}$-module. The map $\varphi : M\times N\to L$ is called A-bilinear if it is $\mathbf{k}$-linear in both arguments and

$$\varphi(ma,n) = \varphi(m,an), \forall a\in A, m\in M, n\in N.$$

Note that if L' is another $\mathbf{k}$-module and $\psi : L \to L'$ an $\mathbf{k}$-linear, then $\psi \circ \varphi : M \times N \to L'$ is A-bilinear.

Here is the universal property we want from the $\mathbf{k}$-module $M \otimes_A N$:

(*) There is an A-bilinear map $M \times N \to M \otimes_A N$ to be denoted by $(m, n) \mapsto m \otimes n$ such that for every A-bilinear map $\varphi : M \times N \to L$ there is a unique $\mathbf{k}$-linear map $\psi : M \otimes_A N \to L$ such that $\varphi(m, n) = \psi(m \otimes n)$, i.e., the following diagram is commutative.



Note that (*) guarantees that $M \otimes_A N$ is unique (if it exists) in the following sense.

Exercise 1.22. If we have another tensor product $M \otimes'_A N$ with bilinear map $(m, n) \mapsto m \otimes' n$ then there is a unique isomorphism $\iota : M \otimes_A N \xrightarrow{\sim} M \otimes'_A N$ satisfying $\iota(m \otimes n) = m \otimes' n$ for all $m \in M, n \in N$.

The tensor product $M \otimes_A N$ also has the following corollary.

Corollary 1.23. *Assume the module $M \otimes_A N$ and the bilinear map $(m, n) \mapsto m \otimes n$ satisfy (*). Then the elements $m \otimes n$ for $m \in M, n \in N$ span the $\mathbf{k}$-module $M \otimes_A N$.*

Proof. Let Q denote the $\mathbf{k}$-submodule of $M \otimes_A N$ spanned by the elements of the form $m \otimes n$. Consider the quotient $L := (M \otimes_A N)/Q$ with epimorphism $\varpi : M \otimes_A N \twoheadrightarrow L$. Note that $\varphi : M \times N \to L, (m, n) \mapsto \varpi(m \otimes n)$ is an A-bilinear map. The corresponding $\mathbf{k}$-linear map $M \otimes_A N \to L$ is ϖ. Note, however, that $\varphi = 0$. So, by the uniqueness property in (*), $\varpi = 0$. On the other hand, ϖ is surjective. It follows that $L = \{0\}$, equivalently, the elements $m \otimes n$ span the $\mathbf{k}$-module $M \otimes_A N$. □

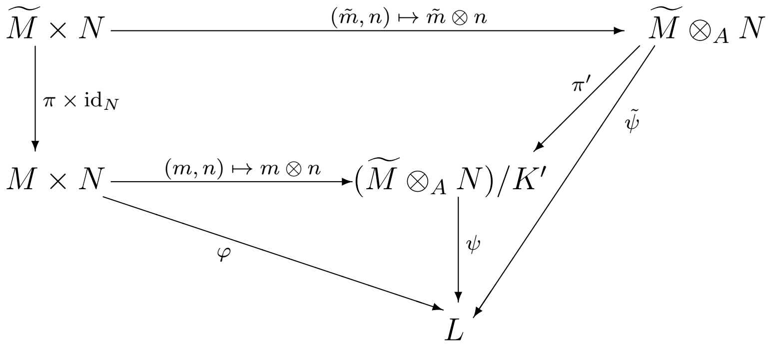
Theorem 1.24. *Tensor product $M \otimes_A N$ exists for all right A-modules M and left A-modules N.*

Proof. Assume, first, that M is a free right A-module with basis e_i, where i is in an index set $\mathcal{I}$. So every element of M is uniquely written as a sum of the form $\sum_{i \in \mathcal{I}} e_i a_i$ with $a_i \in A$, where only finitely many elements a_i are nonzero. Define $M \otimes_A N$ as $N^{\oplus \mathcal{I}}$ and the map $\otimes$ by $(\sum_i e_i a_i) \otimes n := (a_i n)_{i \in \mathcal{I}}$. It is an exercise to check that this map satisfies (*).

For general M, we can represent M as a quotient of a free module. So suppose $M = \widetilde{M}/K$ and $\widetilde{M} \otimes_A N$ exists (for example, for $\widetilde{M}$ we can take a free module with an epimorphism to M). Define a $\mathbf{k}$-submodule K' of $\widetilde{M} \otimes_A N$ as the $\mathbf{k}$-linear span of the elements of the form $k \otimes n$ for $k \in K, n \in N$. We claim that we can take $(\widetilde{M} \otimes_A N)/K'$ for $M \otimes_A N$. First, we produce an A-bilinear map $M \times N \to (\widetilde{M} \otimes_A N)/K', (m, n) \mapsto m \otimes n$. Let $\pi : \widetilde{M} \twoheadrightarrow M$ and $\pi' : \widetilde{M} \otimes_A N \twoheadrightarrow (\widetilde{M} \otimes_A N)/K'$ be the projections. For an element $m \in M$ choose an element $\tilde{m} \in \widetilde{M}$ with $\pi(\tilde{m}) = m$. Set $m \otimes n = \pi'(\tilde{m} \otimes n)$. It is easy to see that this map is well-defined, i.e., independent of the choice of $\tilde{m}$, and is A-bilinear. This is left as an exercise.

Now we need to show that the module $(\widetilde{M} \otimes_A N)/K'$ and $(m, n) \mapsto m \otimes n$ satisfy the universal property (*). Let $\varphi : M \times N \to L$ be an A-bilinear map. Define $\tilde{\varphi} : \widetilde{M} \times N \to L$

as $(\tilde{m}, n) \mapsto \varphi(\pi(\tilde{m}), n)$. This is an A-bilinear map. So we have a unique $\mathbf{k}$-linear map $\tilde{\psi} : \widetilde{M} \otimes_A N \to L$ satisfying $\tilde{\psi}(\tilde{m} \otimes n) = \tilde{\varphi}(\tilde{m}, n) = \varphi(\pi(\tilde{m}), n)$. Note that $\tilde{\psi}(k \otimes n) = \varphi(\pi(k), n) = \varphi(0, n) = 0$. It follows that $\tilde{\psi}$ vanishes on K' and hence factors through a unique map $\psi : (\widetilde{M} \otimes_A N)/K' \to L$. By the construction, we have the following commutative diagram:



We conclude that the map ψ in (*) exists. It is unique because it must make the above diagram commutative and π' is surjective. This finishes the proof of the theorem. □

Example 1.25. Let M, N be free right and left A-modules with bases $e_i, i \in \mathcal{I}$, and $f_j, j \in \mathcal{J}$, respectively. Then $M \otimes_A N$ is a free $\mathbf{k}$-module with basis $e_i \otimes f_j$.

Exercise 1.26. Let $M = A/I$, where I is a right ideal in A. Then $M \otimes_A N = N/IN$. In particular, if $N = A/J$ for a left ideal $J \subset A$, then $M \otimes_A N = A/(I + J)$.

Now suppose that B is another $\mathbf{k}$-algebra, and M is a B-A-bimodule. We claim that $M \otimes_A N$ is a left B-module in a natural way. Namely, fix $b \in B$ and define the operator of multiplication by b on $M \otimes_A N$. For this, note that the map $\varphi_b : M \times N \to M \otimes_A N, (m, n) \mapsto (bm) \otimes n$ is bilinear. So, by (*), there is a unique $\mathbf{k}$-linear map $\psi_b : M \otimes_A N \to M \otimes_A N$ such that $\psi_b(m \otimes n) = (bm) \otimes n$.

Exercise 1.27. Using the uniqueness in (*), show that the assignment $bx := \psi_b(x)$ for $b \in B$ defines a left B-module structure on the $\mathbf{k}$-module $M \otimes_A N$.

It turns out that the resulting B-module $M \otimes_A N$ has a universal property similar to (*).

Definition 1.28. Let L be a B-module. The map $\varphi : M \times N \to L$ is called B-A-bilinear if it is $\mathbf{k}$-linear in both arguments and

$$\varphi(ma, n) = \varphi(m, an), \varphi(bm, n) = b\varphi(m, n), \forall a \in A, b \in B, m \in M, n \in N.$$

Proposition 1.29. *The left B-module $M \otimes_A N$ has the following universal property: let L be a left B-module, $\varphi : M \times N \to L$ be a B-A-bilinear map. Then there is a unique B-linear map $\psi : M \otimes_A N \to L$ such that $\psi(m \otimes n) = \varphi(m, n)$.*

Proof. Every left B-module is also a $\mathbf{k}$-module. By (*), we have a unique $\mathbf{k}$-linear map $\psi : M \otimes_A N \to L$ such that $\psi(m \otimes n) = \varphi(m, n)$. We just need to check that it is B-linear: $\psi(bx) = b\psi(x)$ for all $b \in B, x \in M \otimes_A N$. Thanks to Corollary 1.23, it is enough to prove this for x of the form $m \otimes n$. We have

$$\psi(b(m \otimes n)) = \psi((bm) \otimes n) = \varphi(bm, n) = b\varphi(m, n) = b\psi(m \otimes n).$$

This concludes the proof. □

We finish this section by recalling the tensor product of algebras. Let A_1, A_2 be two $\mathbf{k}$-algebras.

Exercise 1.30. There is a unique $\mathbf{k}$-bilinear product on $A_1 \otimes_{\mathbf{k}} A_2$ such that

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2).$$

It is associative, and $1 \otimes 1$ is a unit.

Similarly, if M_i is a left A_i-module for $i = 1, 2$, then $M_1 \otimes_{\mathbf{k}} M_2$ has a natural structure of a left $A_1 \otimes_{\mathbf{k}} A_2$-module.

We also note that an A-B-bimodule is nothing else but a left $A \otimes_{\mathbf{k}} B^{opp}$-module.

1.6. **Tensor-Hom adjunction.** Let A, B be associative $\mathbf{k}$-algebras, N a B-module, M an A-module, L an A-B-bimodule. As we have pointed out in the previous two sections, $L \otimes_B N$ is an A-module, and $\operatorname{Hom}_A(L, M)$ is a B-module. So it makes sense to consider $\mathbf{k}$-modules

$$\operatorname{Hom}_A(L \otimes_B N, M), \operatorname{Hom}_B(N, \operatorname{Hom}_A(L, M)).$$

The following important result is known as the *tensor-Hom adjunction*.

Theorem 1.31. *We have a natural* $\mathbf{k}$*-linear isomorphism*

$$\operatorname{Hom}_A(L \otimes_B N, M) \xrightarrow{\sim} \operatorname{Hom}_B(N, \operatorname{Hom}_A(L, M)).$$

Proof. By the universal property of the tensor product of a bimodule and a left module, Proposition 1.29, we have a natural isomorphism $\operatorname{Hom}_A(L \otimes_B N, M) \xrightarrow{\sim} \operatorname{Bilin}_{A,B}(L \times N, M)$, where the target is the set of A-B-bilinear maps $L \times N \to M$ with its natural $\mathbf{k}$-module structure. It remains to establish a natural isomorphism $\operatorname{Hom}_B(N, \operatorname{Hom}_A(L, M)) \xrightarrow{\sim} \operatorname{Bilin}_{A,B}(L \times N, M)$. Namely, we send $f \in \operatorname{Hom}_B(N, \operatorname{Hom}_A(L, M))$ to $\psi_f \in \operatorname{Bilin}_{A,B}(L \times N, M)$ given by

$$\psi_f(\ell, n) = [f(n)](\ell).$$

The inverse map sends $\psi \in \operatorname{Bilin}_{A,B}(L \times N, M)$ to $n \mapsto \psi(?, n)$. These two maps are clearly inverse to each other. □

We will need some special cases.

Suppose, first, that we have an algebra homomorphism $B \to A$. Then we can view A as an A-B-bimodule.

Note that $\operatorname{Hom}_A(A, M)$ is naturally identified with M (as an A-module and hence as a B-module). So we have a natural isomorphism

$$\operatorname{Hom}_A(A \otimes_B N, M) \xrightarrow{\sim} \operatorname{Hom}_B(N, M). \tag{1.1}$$

Definition 1.32. The A-module $A \otimes_B N$ is said to be *induced* from N.

Now suppose we have an algebra homomorphism $A \to B$. Then we can view B as an A-B-bimodule. Take this bimodule for L. Note that $B \otimes_B N$ is N viewed as an A-module. So we have a natural homomorphism

$$\operatorname{Hom}_A(N, M) \xrightarrow{\sim} \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, M)). \tag{1.2}$$

Definition 1.33. The B-module $\operatorname{Hom}_A(B, M)$ is said to be *coinduced* from M.

Remark 1.34. Theorem 1.31 and isomorphisms (1.1) and (1.2) can be stated using the language of adjoint functors (hence "adjunction" in the title of the section). Theorem 1.31 implies that the functor $L \otimes_B \bullet : B\text{-Mod} \to A\text{-Mod}$ is left adjoint to $\operatorname{Hom}_A(L, \bullet) : A\text{-Mod} \to B\text{-Mod}$. For a homomorphism $B \to A$ we can talk about the forgetful functor $A\text{-Mod} \to$

B-Mod. The induction functor $A\otimes_B \bullet : B\text{-Mod}\to A\text{-Mod}$ is left adjoint to the forgetful functor. Similarly, for a homomorphism $A\to B$, the coinduction functor $\operatorname{Hom}_A(B,\bullet): A\text{-Mod}\to B\text{-Mod}$ is right adjoint to the forgetful functor $B\text{-Mod}\to A\text{-Mod}$.

2. Completely reducible representations of associative algebras

2.1. **Basic definitions, examples and properties.** Let $\mathbb{F}$ be a field, A be an associative $\mathbb{F}$-algebra, and U be an A-module.

Definition 2.1. We say that U is *irreducible* if it has exactly two distinct submodules: $\{0\}$ and U.

In particular, $\{0\}$ is not irreducible.

Definition 2.2. We say that U is *completely reducible* if for any submodule $U'\subset U$ there exists a submodule $U''\subset U$ (called *complementary* to U') such that $U=U'\oplus U''$.

In particular, every irreducible module is completely reducible (a linguistical paradox!).

Exercise 2.3. Any submodule and any quotient module of a completely reducible module are completely reducible.

Example 2.4. Let U be a finite dimensional vector space over $\mathbb{F}$. Set $A:=\operatorname{End}_{\mathbb{F}}(U)$. Recall that U can be viewed as an A-module. This module is irreducible because for any two nonzero elements $u,v\in U$ there is an linear operator $\alpha: U\to U$ with $\alpha(u)=v$.

We will need two basic properties of irreducible and completely reducible modules.

Proposition 2.5. *Let U_1,U_2 be two completely reducible A-modules. Then $U=U_1\oplus U_2$ is completely reducible.*

Proof. Let $U'\subset U_1\oplus U_2$ be a submodule. We can assume that $U_1\cap U'=\{0\}$. Indeed, otherwise we can find a complementary submodule $U_1''\subset U_1$ to $U_1\cap U'$. By Exercise 2.3, U_1'' is completely reducible. We replace U with $U_1''\oplus U_2$ and U' with $U'\cap(U_1''\oplus U_2)$. A complement to $U'\cap(U_1'\oplus U_2)$ in $U_1'\oplus U_2$ is also a complement to U' in $U_1\oplus U_2$ (the proof of this claim is left as an exercise for the reader). And $[U'\cap(U_1''\oplus U_2)]\cap U_1''=\{0\}$ by the construction. So we arrive at the situation when the intersection is zero.

Now we reduce to the case when $U_1+U'=U_1\oplus U_2$: replace U_2 with its submodule $(U'+U_1)/U_1$. The details are left as an exercise. Then $U'\oplus U_1=U$ and U_1 is a complementary submodule to U'. □

The previous proposition has the following important corollary.

Corollary 2.6. *Let U be a finite dimensional A-module. Then the following two conditions are equivalent:*

- *U is completely reducible.*
- *U is isomorphic to the direct sum of irreducible modules.*

We will also need the following exercise.

Exercise 2.7. Every irreducible A-module is isomorphic to a quotient module of the regular module A. In particular, every irreducible module over a finite dimensional associative $\mathbb{F}$-algebra is finite dimensional.

2.2. **Schur lemma and consequences.** The following important result is known as the Schur lemma.

Theorem 2.8. *Let A be an associative $\mathbb{F}$-algebra and let U, V be irreducible A-modules. Then the following claims are true.*

(1) *If U, V are non-isomorphic, then $\mathrm{Hom}_A(U, V) = 0$.*
(2) *$\mathrm{End}_A(U)$ is a skew-field. In particular, if U is finite dimensional, and $\mathbb{F}$ is algebraically closed, then $\mathrm{End}_A(U)$ consists of scalar maps, so, $\dim \mathrm{End}_A(U) = 1$.*

Proof. (1): let $\varphi : U \to V$ be a homomorphism. Assume it is nonzero. Then $\ker\varphi \subset U, \mathrm{im}\,\varphi \subset V$ are submodules. Moreover, $\ker\varphi \neq U, \mathrm{im}\,\varphi \neq \{0\}$. Since U, V are irreducible, it follows that $\ker\varphi = \{0\}, \mathrm{im}\,\varphi = V$, i.e., φ is an isomorphism. We arrive at a contradiction with the condition that U, V are non-isomorphic, which implies $\varphi = 0$.

(2): let $\varphi \in \mathrm{End}_A(U)$. Assume $\varphi \neq 0$. Arguing as in the previous part, we see that $\ker\varphi = \{0\}, \mathrm{im}\,\varphi = U$. So φ is an isomorphism. It follows that φ is invertible in $\mathrm{End}_A(U)$.

If U is finite dimensional and $\mathbb{F}$ is algebraically closed, then φ has an eigenvalue, say $z \in \mathbb{F}$. The element $\varphi - z\,\mathrm{id}_U$ is not invertible, hence is zero. □

Definition 2.9. Let U be an A-module. We say U is *endo-trivial* if $\mathrm{End}_A(U)$ consists of scalar maps.

We note that (2) of Theorem 2.8 can be generalized to infinite dimensional irreducible representations under some assumptions. For example, we have the following result whose proof we omit.

Proposition 2.10. *Suppose that $\mathbb{F}$ is algebraically closed and uncountable (e.g., $\mathbb{F} = \mathbb{C}$) and A has countable dimension over $\mathbb{F}$. Let U be an irreducible A-module. Then U is endo-trivial.*

Theorem 2.8 allows to describe the action of the *center* of A on an irreducible module.

Definition 2.11. By the center of A we mean $Z(A) := \{z \in A | za = az, \forall a \in A\}$.

This is a commutative algebra. Note that any element of the center acts on any A-module by an A-module endomorphism.

Exercise 2.12. Let U be an endo-trivial A-module (for example, a finite dimensional irreducible module in the case when $\mathbb{F}$ is algebraically closed, see Theorem 2.8). Let $z \in Z(A)$. Then z acts on U by a scalar. The assignment sending z to that scalar is an algebra homomorphism $Z(A) \to \mathbb{F}$.

This algebra homomorphism is called the *central character* of U.

2.3. **Completely reducible representations.** Below we will use the Schur lemma to study the (finite) direct sums of endo-trivial irreducible modules (for example, completely reducible finite dimensional modules in the case when the field is algebraically closed). We notice that such a module can be written as

$$\bigoplus_{i=1}^{k} U_i \otimes M_i, \tag{2.1}$$

where $U_1, \ldots, U_k$ are endo-trivial irreducible modules and $M_1, \ldots, M_k$ are vector spaces, and the action of A is given by

$$a(u_1 \otimes m_1, \ldots, u_k \otimes m_k) = ([au_1] \otimes m_1, \ldots, [au_k] \otimes m_k). \tag{2.2}$$

(2.1) follows from the observation that $U_1^{\oplus \ell}$ is identified with $U_1 \otimes \mathbb{F}^{\ell}$.

Definition 2.13. Below we will call M_i the *multiplicity space* (of U_i).

Theorem 2.8 allows to compute the space of homomorphisms between modules of the form (2.1). Pick multiplicity spaces $M_1^1,\ldots,M_k^1,M_1^2,\ldots,M_k^2$ and form the direct sums $U^j:=\bigoplus_{i=1}^k U_i\otimes M_i^j$ for $j=1,2$.

Now we produce a linear map

$$\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^2)\to \operatorname{Hom}_A(U^1,U^2).$$

Let $\underline{\varphi}:=(\varphi_1,\ldots,\varphi_k)$ be a typical element of $\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^2)$. Define a map $\psi_{\underline{\varphi}}:U^1\to U^2$ by

$$\psi_{\underline{\varphi}}(\sum_{i=1}^k u_i\otimes m_i^1):=\sum_{i=1}^k u_i\otimes \varphi_i(m_i^1),\quad u_i\in U_i, m_i^1\in M_i^1. \tag{2.3}$$

This is a well-defined linear map. It follows from (2.2) that it is A-linear.

Theorem 2.14. *The following claims are true.*

(1) *The map $\underline{\varphi}\mapsto\psi_{\underline{\varphi}}$ defines a vector space isomorphism*

$$\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^2)\xrightarrow{\sim} \operatorname{Hom}_A(U^1,U^2).$$

(2) *Every A-module homomorphism $U^1\to U^2$ sends $U_i\otimes M_i^1$ to $U_i\otimes M_i^2$ for all i.*

Proof. (1): Note that $\psi_{\underline{\varphi}}=0$ implies $\underline{\varphi}=0$. So to prove that the map $\underline{\varphi}\mapsto\psi_{\underline{\varphi}}$ is an isomorphism we need to show that $\dim\operatorname{Hom}_A(U^1,U^2)=\dim\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^2)$. The latter follows from Theorem 2.8 and is left as an exercise.

(2): To show that every A-linear map $U^1\to U^2$ sends $U_i\otimes M_i^1$ to $U_i\otimes M_i^2$, notice that the maps of the form $\psi_{\underline{\varphi}}$ have this property by definition. Since $\underline{\varphi}\mapsto\psi_{\underline{\varphi}}$, we are done. □

We note that (2) can also be deduced from the Schur lemma directly, left as an exercise.

The following remark is important for what follows.

Remark 2.15. This remark describes the compatibility of the isomorphism in Theorem 2.14 with taking compositions. Let M_i^1,M_i^2,M_i^3 with $i=1,\ldots,k$, be three collections of multiplicity spaces. Let $U^j:=\bigoplus_{i=1}^k U_i\otimes M_i^j, j=1,2,3$, be the corresponding A-modules. So we have the composition map

$$\operatorname{Hom}_A(U^2,U^3)\times\operatorname{Hom}_A(U^1,U^2)\to\operatorname{Hom}_A(U^1,U^3).$$

Similarly, we have the componentwise composition map

$$\left(\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^2,M_i^3)\right)\times\left(\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^2)\right)\to\bigoplus_{i=1}^k \operatorname{Hom}_{\mathbb{F}}(M_i^1,M_i^3).$$

The maps $\psi_\bullet$ defined by (2.3) intertwine the composition maps meaning that

$$\psi_{\underline{\varphi}_2\circ\underline{\varphi}_1}=\psi_{\underline{\varphi}_2}\circ\psi_{\underline{\varphi}_1}.$$

In particular, consider the situation when $M_i^1 = M_i^2 = M_i^3$ for all i. The spaces $\bigoplus_{i=1}^k \operatorname{Hom}_A(M_i^1, M_i^1)$ and $\operatorname{Hom}_A(U^1, U^1)$ have natural algebra structures, where the multiplication is given by composition. The previous paragraph implies that $\underline{\varphi} \mapsto \psi_{\underline{\varphi}}$ is an algebra isomorphism.

Now we record another important corollary of Theorem 2.14. Let U be of the form (2.1).

Corollary 2.16. *We have an identification* $\operatorname{Hom}_A(U_i, U) \xrightarrow{\sim} M_i$. *Moreover, the isomorphism* $\bigoplus_{i=1}^k U_i \otimes \operatorname{Hom}_A(U_i, U) \xrightarrow{\sim} U$ *is given by*

$$\sum_{i=1}^k u_i \otimes \varphi_i \mapsto \sum_{i=1}^k \varphi_i(u_i). \tag{2.4}$$

Proof. In Theorem 2.14 we take $M_\ell^1 = \mathbb{F}^{\oplus \delta_{i\ell}}$ and $M_2^i = M_i$. Then we use the natural identification $\operatorname{Hom}_{\mathbb{F}}(\mathbb{F}, M_i) \xrightarrow{\sim} M_i$ to arrive at the identification $\operatorname{Hom}_A(U_i, U) \xrightarrow{\sim} M_i$. Under the identifications of the source and the target of (2.4) with $\bigoplus_{i=1}^k U_i \otimes M_i$, that map becomes the identity. □

Now we proceed to describing submodules of direct sums of irreducible endo-trivial modules. Let $U_1, \ldots, U_k$ be irreducible endo-trivial A-modules and $M_1, \ldots, M_k$ be finite dimensional vector spaces. Consider the A-module $U := \bigoplus_{i=1}^k U_i \otimes M_i$.

Proposition 2.17. *For any A-submodule $U' \subset U$, there are uniquely determined subspaces $M_i' \subset M_i$ such that $U' = \bigoplus_{i=1}^k U_i \otimes M_i'$ as a submodule of U.*

Proof. Recall, Section 1.4, that $\operatorname{Hom}_A(U_i, U') \hookrightarrow \operatorname{Hom}_A(U_i, U)$. We set $M_i' := \operatorname{Hom}_A(U_i, U')$. We need to show that U' coincides with $\bigoplus_{i=1}^k U_i \otimes M_i'$ as a submodule of U.

The A-module U' is completely reducible, as a submodule of a completely reducible module, see Corollary 2.6. Corollary 2.16 yields an A-module isomorphism $\bigoplus_{i=1}^k U_i \otimes M_i' \xrightarrow{\sim} U'$, denote it by α. What we need to show is that α intertwines the embeddings $U' \hookrightarrow U$ and

$$\bigoplus_{i=1}^k U_i \otimes M_i' \hookrightarrow \bigoplus_{i=1}^k U_i \otimes M_i (= U). \tag{2.5}$$

Let ι be the embedding $U' \hookrightarrow U$ so that the embedding $M_i' \hookrightarrow M_i$ is given by

$$\varphi_i \mapsto \iota \circ \varphi_i. \tag{2.6}$$

Recall that α is given by (2.4). So $\iota \circ \alpha$ is given by

$$\sum_{i=1}^k u_i \otimes \varphi_i \mapsto \sum_{i=1}^k (\iota \circ \varphi_i)(u_i).$$

Using (2.6) we see that $\iota \circ \alpha$ coincides with the (2.5). □

Here is an interesting corollary of Proposition 2.17. Let $U_1, \ldots, U_k$ be pairwise non-isomorphic endo-trivial finite dimensional irreducible A-modules. The A-module structure on U_i yields an algebra homomorphism $\beta_i : A \to \operatorname{End}_{\mathbb{F}}(U_i)$. Let $\beta : A \to \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$ denote the direct sum of this homomorphisms, this is an algebra homomorphism.

Theorem 2.18. *The homomorphism β is surjective.*

Proof. Replacing A with $A/\ker\beta$ we can assume that β is injective. The homomorphism β equips $\bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i)$ with an A-bimodule structure so that A is embedded as a subbimodule. We can view $\bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i)$ as a left and as a right A-module and A is a submodule for both these structures. We have a natural isomorphism $\mathrm{End}_{\mathbb{F}}(U_i) \cong U_i \otimes U_i^*$. When we view $\mathrm{End}_{\mathbb{F}}(U_i)$ as a left A-module, then U_i is an irreducible (left) A-module and U_i^* is its multiplicity space. When we view $\mathrm{End}_{\mathbb{F}}(U_i)$ as a right A-module, U_i^* is an irreducible (right) A-module, and U_i is its multiplicity space. Applying Proposition 2.17 to the left A-module structure on $\bigoplus_{i=1}^k U_i \otimes U_i^*$, we see that $A = \bigoplus_{i=1}^k U_i \otimes V_i$ for a uniquely determined subspace $V_i \subset U_i^*$. Similarly, for the right module structure we get $A = \bigoplus_{i=1}^k W_i \otimes U_i^*$ for $W_i \subset U_i$. So $U_i \otimes V_i^* = W_i \otimes U_i^*$ as subspaces in $U_i \otimes U_i^*$ for all i. This is only possible if $V_i = U_i^*, W_i = U_i$ or $V_i = \{0\}, W_i = \{0\}$. However the former is impossible because the element $1 \in A$ goes to $(\mathrm{id}_{U_i})_{i=1}^k$. We conclude that $\mathrm{End}_{\mathbb{F}}(U_i) \subset \mathrm{im}\,\beta$ for all i, equivalently, β is surjective. □

Corollary 2.19. *Let $\mathbb{F}$ be an algebraically closed field and A be a finite dimensional $\mathbb{F}$-algebra. Then the set of isomorphism classes of irreducible A-modules is finite and nonempty.*

Proof. The set is nonempty because A has a nonzero representation, e.g., A itself, that must have an irreducible subrepresentation. Any irreducible representation is finite dimensional by Exercise 2.7. It is endo-trivial by (2) of Theorem 2.8. So we can apply Theorem 2.18 to see that for any collection $U_1, \ldots, U_k$ of pairwise non-isomorphic irreducible A-modules, the homomorphism $A \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{F}}(U_i)$ is surjective, hence $\sum_{i=1}^k (\dim U_i)^2 \leqslant \dim A$. This implies that the number of isomorphism classes of irreducibles is finite. □

2.4. **Simple algebras.** In Representation theory, a special role is played by objects that are *simple*.

Definition 2.20. An associative algebra A is called *simple* if the only two-sided ideals in A are $\{0\}$ and A, i.e., if it is irreducible as a bimodule over itself.

Theorem 2.21. *Let $\mathbb{F}$ be an algebraically closed field and A be a finite dimensional $\mathbb{F}$-algebra. Then the following two conditions are equivalent:*

(1) *A is simple,*
(2) *$A \cong \mathrm{End}_{\mathbb{F}}(U)$ for a finite dimensional vector space U.*

Proof. (1)⇒(2). The algebra A has an irreducible representation, say U. This gives an algebra homomorphism $A \to \mathrm{End}_{\mathbb{F}}(U)$. Since A is simple, this homomorphism is injective. It is surjective by Theorem 2.18. So it is an isomorphism.

(2)⇒(1). Let I be a two-sided ideal (equivalently, subbimodule) in $\mathrm{End}_{\mathbb{F}}(U)$. Arguing as in the proof of Theorem 2.18, we see that there are subspaces $V \subset U^*, W \subset U$ such that $I = W \otimes U^* = U \otimes V \subset U \otimes U^* = \mathrm{End}(U)$. And similarly to that proof, we see that $I = \{0\}$ or $\mathrm{End}_{\mathbb{F}}(U)$. □

We can completely describe the finite dimensional representations of $\mathrm{End}_{\mathbb{F}}(U)$. Note that U is an irreducible representation of $\mathrm{End}_{\mathbb{F}}(U)$. Assume $\mathbb{F}$ is a general field.

Theorem 2.22. *Every finite dimensional module V for $A := \mathrm{End}_{\mathbb{F}}(U)$ is isomorphic to the direct sum of several copies of U.*

Proof. Recall that every finitely generated A-module is the quotient of $A^{\oplus \ell}$ for a suitable integer $\ell > 0$. But as an A-module, we have $A = U \otimes U^*$ with U^* being the multiplicity space. So we have an A-module epimorphism $\pi : U \otimes M \twoheadrightarrow V$ for a suitable vector space

M. The representation U is endo-trivial, so we can apply Proposition 2.17 to $\ker \pi$ getting $\ker \pi = U \otimes M_0$. Then $V \cong (U \otimes M)/(U \otimes M_0) \cong U \otimes (M/M_0)$ and we are done. □

2.5. **Semisimple algebras.**

Definition 2.23. A finite dimensional $\mathbb{F}$-algebra is called *semisimple* if it is isomorphic to the direct sum of simple algebras.

So if $\mathbb{F}$ is algebraically closed, then the semisimple algebras are the same thing as the direct sums of matrix algebras.

We will start by describing the representations of direct sums of matrix algebras.

Theorem 2.24. *Let $U_1, \ldots, U_k$ be finite dimensional vector spaces over $\mathbb{F}$. Set*

$$A := \bigoplus_{i=1}^{k} \operatorname{End}_{\mathbb{F}}(U_i)$$

so that each U_i is an irreducible A-module. Every irreducible A-module is isomorphic to exactly one of the U_i's. Every finite dimensional A-module V is isomorphic to the direct sum of several copies of $U_1, \ldots, U_k$.

This theorem can be proved in the same way as its special case, Theorem 2.22. Alternatively, one can deduce Theorem 2.24 from Theorem 2.22 and the following lemma. Namely, let $A_1, \ldots, A_k$ be associative algebras and set $A := \bigoplus_{i=1}^{k} A_i$. Let ϵ_i denote the unit in A_i for $i = 1, \ldots, k$. Note that $\sum_{i=1}^{k} \epsilon_i$ is the unit in A.

Lemma 2.25. *Let M be an A-module. Set $M_i := \epsilon_i M$. Then the following claims hold:*

1. *We have $M = \bigoplus_{i=1}^{k} M_i$, the direct sum of vector spaces.*
2. *The subspace M_i is an A-submodule. Moreover, the action of A on M_i factors through the action of A_i under the natural epimorphism $A \mapsto A_i$. So, $M = \bigoplus_{i=1}^{k} M_i$ is the decomposition into the direct sum of A-modules.*

Note that this lemma reduces Theorem 2.24 to Theorem 2.22.

Proof. Proof of (1): for $m \in M$, we have

$$m = 1m = (\epsilon_1 + \ldots + \epsilon_k)m = \sum_{i=1}^{k} \epsilon_i m. \tag{2.7}$$

This shows that every element $m \in M$ can be decomposed as the sum of elements of M_i. Note that $\epsilon_i \epsilon_j = \delta_{ij} \epsilon_i$. Hence ϵ_i acts by 1 on M_i and by 0 on M_j. This observation implies that if $m = \sum_i m_i$ with $m_i \in M_i$, then $m_i = \epsilon_i m$. (1) follows.

Proof of (2): note that $a\epsilon_i = \epsilon_i a$. It follows that M_i is an A-submodule of M. Also the projection $A \twoheadrightarrow A_i$ can be written as $a \mapsto \epsilon_i a$. It follows that the A-action on M_i factors through this projection. This finishes the proof of (2). □

In what follows we will need a corollary of the theorem.

Corollary 2.26. *Let $\mathbb{F}$ be an algebraically closed field, and A be a semisimple finite dimensional associative algebra over $\mathbb{F}$. Then the following claims hold:*

1. *The number of isomorphism classes of irreducible A-modules equals to the dimension of the center of A.*

(2) *Different irreducible modules have different central characters (see the end of Section 2.2 for definition).*

Proof. Recall that $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. By Theorem 2.24, the irreducibles are precisely $U_1, \ldots, U_k$, so the number of isomorphism classes equals k. On the other hand, for associative algebras $A_1, \ldots, A_k$, we have $Z(\bigoplus_{i=1}^k A_i) = \bigoplus_{i=1}^k Z(A_i)$. The center of $\operatorname{End}_{\mathbb{F}}(U_i)$ consists of scalar endomorphisms and hence is one-dimensional. So $\dim Z(A) = k$, and (1) follows. To prove (2) we note that $Z(A)$ acts on U_i via the projection $Z(A) \twoheadrightarrow Z(A_i) \cong \mathbb{F}$. For different i's, these projections are different (for example, because they have different kernels). □

Now we discuss the equivalent characterizations of semisimple algebras.

Let A be a finite dimensional algebra. We say that a 2-sided ideal $I \subset A$ is *nilpotent* if $I^n = \{0\}$ for some $n > 0$. An easy exercise is to show that if $I, J \subset A$ are nilpotent ideals, then $I + J$ is nilpotent as well (if $I^n = J^n = \{0\}$, then $(I + J)^{2n} = \{0\}$). It follows that there is the unique maximal nilpotent ideal.

Definition 2.27. The maximal nilpotent ideal of A is called the *radical* of A and is denoted by $\operatorname{rad}(A)$.

The following theorem provides equivalent characterizations of semisimple algebras.

Theorem 2.28. *Suppose $\mathbb{F}$ is algebraically closed. Let A be a finite dimensional associative algebra over $\mathbb{F}$. The following conditions are equivalent:*

(a) *A is semisimple,*
(b) *all finite dimensional representations of A are completely reducible,*
(c) $\operatorname{rad}(A) = \{0\}$.

Proof. (a)⇒(b) is Theorem 2.24.

(b)⇒(c): Let $I := \operatorname{rad}(A)$. We have $I^n := \{0\}$ for some $n > 0$. Let N be a finite dimensional A-module. Then $I^\ell N$ is an A-submodule for all $\ell = 0, \ldots, n$. Then for all ℓ we can find an A-submodule N_ℓ with $I^\ell N = N_\ell \oplus I^{\ell+1} N$. Since $I N_\ell \subset I^{\ell+1} N$, we see that $I N_\ell = \{0\}$. But $N = N_0 \oplus N_1 \oplus \ldots \oplus N_{n-1}$. It follows that $IN = \{0\}$. Applying this to $N := A$, the regular module, we see that $I = \{0\}$, which is (c).

(c)⇒(a): Let $N_1, \ldots, N_k$ be all pairwise non-isomorphic irreducible A-modules, see Corollary 2.19. So we have an algebra epimorphism $A \twoheadrightarrow \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(N_i)$, Theorem 2.18. Let I be the kernel. We claim that it is a nilpotent ideal. Note that, by the construction, I acts by 0 on every irreducible A-module. Consider the regular A-module A. As any finite dimensional A-module it admits a filtration $A = A_0 \supset A_1 \supset A_2 \supset \ldots \supset A_n = \{0\}$ such that A_i/A_{i+1} is irreducible for all I. Since I acts by 0 on every irreducible A-module, we see that $IA_i \subset A_{i+1}$ for all i. It follows that $I^n = \{0\}$. Since $\operatorname{rad}(A) = \{0\}$, we see that $I = \{0\}$. □

In the case when $\operatorname{char} \mathbb{F} = 0$, there is yet another equivalent characterization of semisimple algebras. For an A-module U we can consider the following bilinear form on A: $(a, b)_U := \operatorname{tr}_U(ab)$. This form is symmetric. In particular, we can take $U := A$.

Theorem 2.29. *Assume that $\operatorname{char} \mathbb{F} = 0$. Let A be a finite dimensional $\mathbb{F}$-algebra. Then A is semisimple if and only if the form $(\cdot, \cdot)_A$ is non-degenerate.*

Proof. Suppose A is semisimple, i.e., $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. The restriction of $(\cdot, \cdot)_A$ to the direct summand $\operatorname{End}_{\mathbb{F}}(U_i)$ coincides with the form $(\cdot, \cdot)_{\operatorname{End}_{\mathbb{F}}(U_i)}$. So we need to show that the latter is nondegenerate. Let $E_{j\ell}$ denote the (j, ℓ) matrix unit in $\operatorname{End}_{\mathbb{F}}(U_i)$. Then we have

$$(E_{j\ell}, E_{j'\ell'})_{\operatorname{End}_{\mathbb{F}}(U_i)} = \delta_{\ell,j'} \operatorname{tr}_{\operatorname{End}_{\mathbb{F}}(U_i)}(E_{j\ell'}) = \delta_{\ell,j'} \delta_{\ell',j} \dim U_i.$$

Since $\operatorname{char}\mathbb{F} = 0$, we see that the basis $E_{j\ell}$ has a dual basis with respect to the form $(\cdot,\cdot)_{\operatorname{End}_{\mathbb{F}}(U_i)}$, this is $(\dim U_i)^{-1}E_{\ell j}$. Note that the different summands of A are orthogonal with respect to $(\cdot,\cdot)_A$. It follows that $(\cdot,\cdot)_A$ is the direct sum of non-degenerate forms, hence is non-degenerate.

Now suppose $(\cdot,\cdot)_A$ is nondegenerate. If I is a nilpotent ideal, then $a^n = 0$, hence $\operatorname{tr}_A(a) = 0$ for all $a \in I$. It follows that I lies in the kernel of the form $(\cdot,\cdot)_A$. So $I = \{0\}$, and A is semisimple by the equivalence (a)$\Leftrightarrow$(c) of Theorem 2.28. □

Finally, we record the following important property known as the double centralizer theorem.

Theorem 2.30. *Let V be a finite dimensional $\mathbb{F}$-vector space and $A \subset \operatorname{End}_{\mathbb{F}}(V)$ be a semisimple subalgebra. Set $B := \operatorname{End}_A(V)$. Then we have $A = \operatorname{End}_B(V)$, the equality of subalgebras in $\operatorname{End}_{\mathbb{F}}(V)$.*

Proof. We have $A = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(U_i)$. The space V is a faithful A-module. By Theorem 2.28, V is completely reducible, so we have an A-linear isomorphism $V \cong \bigoplus_{i=1}^k U_i \otimes M_i$. The way A embeds into $\operatorname{End}_{\mathbb{F}}(V)$ in terms of this decomposition is $(\varphi_1,\ldots,\varphi_k) \mapsto \sum_{i=1}^k \varphi_i \otimes \operatorname{id}_{M_i}$, where the ith summand acts on $U_i \otimes M_i$ only. The multiplicity spaces M_i are all nonzero because V is faithful. By Remark 2.15, $B = \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{F}}(M_i)$. It is embedded into $\operatorname{End}_{\mathbb{F}}(V)$ via $(\psi_1,\ldots,\psi_k) \mapsto \sum_{i=1}^k \operatorname{id}_{U_i} \otimes \psi_i$. If we view V as a B-module, the spaces U_i now play the role of multiplicity spaces. So, by symmetry, $\operatorname{End}_B(V) = A$. □

2.6. **The case of non-closed fields.** Suppose now that $\mathbb{F}$ is an arbitrary field and A is an associative $\mathbb{F}$-algebra. Let U be an irreducible A-module. By Theorem 2.8, $\operatorname{End}_A(U)$ is a skew-field. We denote its opposite skew-field by $\mathbb{H}$ or $\mathbb{H}_U$ when we need to indicate the dependence on U, so that U becomes a right vector space over $\mathbb{H}$. Of course, U is endo-trivial if and only if $\mathbb{H}_U = \mathbb{F}$. Note that we have $U^{\oplus n} \cong U \otimes_{\mathbb{H}} M$ for $M := \mathbb{H}^n$, where we view M as a left vector space over $\mathbb{H}$. Arguing as in the proof of Proposition 2.17, one can show that every A-submodule of $U^{\oplus n}$ has the form $U \otimes_{\mathbb{H}} M'$ for a uniquely determined left subspace $M' \subset M$. This claim generalizes to the case of direct sum of several irreducible modules in a straightforward fashion. Using this we arrive at the generalization of the density theorem, Theorem 2.18:

$$A \twoheadrightarrow \bigoplus_{i=1}^k \operatorname{End}_{\mathbb{H}_{U_i}}(U_i).$$

Using this we see that all finite dimensional simple $\mathbb{F}$-algebras are of the form $\operatorname{End}_{\mathbb{H}}(U)$, where $\mathbb{H}$ is a finite dimensional skew-field over $\mathbb{F}$ and U is a right vector space over $\mathbb{H}$. We have straightforward analogs of Theorems 2.22, 2.24, 2.28, 2.29. Theorem 2.30 holds verbatim. We leave the details to the readers.

3. Representations of finite groups

Throughout the section G denotes a group. For a field $\mathbb{F}$ we can consider the group algebra $\mathbb{F}G$. Recall that a representation of G over $\mathbb{F}$ is the same thing as a representation of the associative algebra $\mathbb{F}G$, Example 1.16.

3.1. **Tensor product, dual and** Hom **representations.** Let U, V be two representations of G. We can equip $U \otimes V (= U \otimes_{\mathbb{F}} V)$ with a unique representation of G satisfying

$$g(u \otimes v) = (gu) \otimes (gv), \forall g \in G, u \in U, v \in V.$$

This is the so called *tensor product representation.*

We can also equip U^* with the structure of a representation of G: there is a unique representation of G in U^* such that

$$\langle g\alpha, u\rangle = \langle \alpha, g^{-1}u\rangle, \forall g \in G, \alpha \in U^*, u \in U.$$

More generally, we can equip $\operatorname{Hom}_{\mathbb{F}}(U, V)$ with the structure of a G-representation:

$$[g\varphi](u) = g[\varphi(g^{-1}u)].$$

For $V = \mathbb{F}$, the trivial representation, we recover the dual representation of G.

The following exercise records some properties of these representations.

Exercise 3.1. The following claims hold.

(1) The tensor product of representations has the usual properties: it is associative, distributive and commutative.
(2) The natural map $U^* \otimes V \to \operatorname{Hom}(U, V)$ is a homomorphism of representations.
(3) The subspace $\operatorname{Hom}_G(U, V) \subset \operatorname{Hom}(U, V)$ of homomorphisms of representations coincides with the subspace of G-invariant elements.

3.2. **Complete reducibility.** In this section G is a finite group. Until the further notice we assume that $\mathbb{F}$ is an algebraically closed field of characteristic 0. The following result is of fundamental importance.

Theorem 3.2. *The algebra $\mathbb{F}G$ is semisimple.*

Proof. Thanks to Theorem 2.29, we need to show that the symmetric bilinear form $(\cdot, \cdot)_{\mathbb{F}G}$ defined before that theorem is non-degenerate. For $g, g' \in G$, we have

$$(g, g')_{\mathbb{F}G} = \operatorname{tr}_{\mathbb{F}G}(gg'). \tag{3.1}$$

On the basis elements $h \in G$, the element gg' acts by $h \mapsto gg'h$. So the right hand side of (3.1) is $\delta_{gg',1}|G|$. Since $\operatorname{char}\mathbb{F} = 0$, the form is non-degenerate: the dual basis of $(g)_{g\in G}$ is $|G|^{-1}(g^{-1})_{g\in G}$. This completes the proof. $\square$

Corollary 3.3. *Recall that $\mathbb{F}$ is algebraically closed of characteristic 0. The following claims are true.*

(1) *Every finite dimensional representation of G is completely reducible.*
(2) *The number of isomorphism classes of irreducible representations is equal to the number of conjugacy classes in G.*
(3) *If $U_1, \ldots, U_k$ are all pairwise non-isomorphic irreducible representationsm then $|G| = \sum_{i=1}^k (\dim U_i)^2$.*

Proof. (1): this follows by combining Theorem 3.2 with the equivalence (a)⇔(b) of Theorem 2.28.

(2): thanks to Corollary 2.26, we need to show that $\dim Z(\mathbb{F}G)$ equals to the number of conjugacy classes in G. This will follow if we check that

$$Z(\mathbb{F}G) = \{\sum_{g\in G} a_g g | a_g \text{ is constant on conjugacy classes}\}. \tag{3.2}$$

Indeed an element $z \in \mathbb{F}G$ is central if and only if $hz = zh$ for all $h \in G$ because the elements h form a basis in $\mathbb{F}G$. The equation $hz = zh$ is equivalent to $z = hzh^{-1}$. If $z = \sum_g a_g g$, then $hzh^{-1} = \sum_g a_g hgh^{-1} = \sum_g a_{h^{-1}gh} g$. (3.2) follows.

(3): this equality follows from two different ways to compute the dimension of the semisimple algebra $\mathbb{F}G$ (the details are left as an exercise). □

Example 3.4. Consider the group S_4 with 24 elements. As in any symmetric group, the conjugacy classes are parameterized by partitions, in this case, of 4. The number of such partitions is equal to 5. So we have 5 irreducible representations. The sum of their dimensions squared equals 24. Now we construct these representations. We have the trivial and sign representations, to be denoted by $\mathrm{triv}_4, \mathrm{sgn}_4$. They are 1-dimensional. Next, S_4 acts on $\mathbb{C}^4$ by permuting the elements of the natural basis (the permutation representation). This representation has the trivial subrepresentation, $\{(x, x, x, x) | x \in \mathbb{C}\}$. It has a unique complement, $\{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 0\}$. This is the so called *reflection* representation of S_4 to be denoted by refl_4, the dimension is 3. There is another three dimensional irreducible representation, $\mathrm{sgn}_3 \otimes \mathrm{refl}_3$ – clearly tensoring with a one-dimensional representation sends every irreducible representation to an irreducible one. The resulting 3-dimensional irreducible representation is not isomorphic to refl_3: indeed, for a reflection $(ij) \in S_4$, its determinant in refl_3 is -1, while the determinant in $\mathrm{sgn}_3 \otimes \mathrm{refl}_3$ is 1. So far, we have constructed four pairwise non-isomorphic irreducible representations of S_4. We have exactly one more and it must have dimension 2. To construct this representation recall that we have an epimorphism $\pi : S_4 \twoheadrightarrow S_3$ with kernel whose nontrivial elements with cycle type $(2, 2)$. The element π is defined on the generators as follows:

$$(1, 2) \mapsto (1, 2), (2, 3) \mapsto (2, 3), (3, 4) \mapsto (1, 2).$$

The reflection representation refl_3 pulls back under π to a representation of S_4. Denote the resulting representation by V_2. It is irreducible because π is surjective and has dimension 2. So we have obtained a complete classification of irreducible representations of S_4.

Exercise 3.5. Let G be finite and abelian. Then all irreducible representations are one-dimensional (and hence there are $|G|$ of them).

Remark 3.6. This remark is important for what follows later in the course. It would be tempting to interpret (2) of Corollary 3.3 as the claim that the irreducible representations of G are *classified* up to isomorphism by the conjugacy classes. Unfortunately, this is not so – there is no natural way to assign a representation to a conjugacy class, in general. So (2) of Corollary 3.3 is a *counting*, not a classification result. For example, consider the case when G is abelian. Let $G^\vee$ denote the set of isomorphism classes of irreducible representations of G. The tensor product of representations equips $G^\vee$ with an associative product, where the trivial representation is the unit. In fact, $G^\vee$ is a group, the inverse corresponds to taking the dual representation. It is not difficult to show that $G^\vee$ is isomorphic to G. However, there is no natural isomorphism. For example, when G is cyclic, $G \cong \mathbb{Z}/n\mathbb{Z}$, then an identification of G and $G^\vee$ amounts to choosing a primitive nth root of 1. And when $G = \mathbb{F}_p^{\oplus n}$, then an identification of G and $G^\vee$ is the same thing as a non-degenerate bilinear form on the $\mathbb{F}_p$-vector space G. There are no natural choices in either of these cases.

3.3. Characters.

Definition 3.7. Let G be a group and U be a finite dimensional representation of G. By the *character* of U we mean the function $\chi_U : G \to \mathbb{F}$ given by

$$\chi_U(g) = \mathrm{tr}_U(g).$$

Exercise 3.8. Show that

(1) χ_U is constant on the conjugacy classes,
(2) $\chi_{U\oplus V} = \chi_U \oplus \chi_V$ (with respect to the pointwise addition),
(3) $\chi_{U\otimes V} = \chi_U \chi_V$ (with respect to the pointwise multiplication).

Until the end of the section we assume that G is finite and $\mathbb{F} = \mathbb{C}$. In particular, every representation of G is completely reducible, Corollary 3.3. We write $\mathbb{C}[G]$ for the algebra of complex valued functions on G (with pointwise multiplication) and $\mathbb{C}[G]^G$ for the subalgebra of all functions that are constant on conjugacy classes (equivalently, invariant under the conjugation action of G on itself). Note that $\chi_U \in \mathbb{C}[G]^G$ for all finite dimensional representations U.

We introduce a hermitian scalar product on $\mathbb{C}[G]^G$ by the following formula

$$(\chi_1, \chi_2) := |G|^{-1} \sum_{g\in G} \chi_1(g)\overline{\chi}_2(g), \tag{3.3}$$

where $\overline{\bullet}$ in the right hand side is the complex conjugation.

We have the following interpretation of the scalar product of characters.

Proposition 3.9. *Let U, V be finite dimensional representations of G. Then*

$$(\chi_U, \chi_V) = \dim \mathrm{Hom}_G(V, U).$$

Proof. First, note that $\chi_{U^*} = \overline{\chi}_U$. Using (3) of Exercise 3.8 combined with the natural (hence G-equivariant) isomorphism $V \otimes U^* \cong \mathrm{Hom}(U, V)$, we see that $\chi_{\mathrm{Hom}(V,U)} = \chi_V \overline{\chi}_U$.

Consider the averaging element $\epsilon := |G|^{-1} \sum_{g\in G} g \in \mathbb{C}G$. Its image in an arbitrary G-representation W is the subspace of invariants W^G. Moreover, it is a projector to W^G meaning, additionally, that $\epsilon|_{W^G} = \mathrm{id}_{W^G}$. It follows that $\mathrm{tr}_W(\epsilon) = \dim W^G$. We apply this to $W = \mathrm{Hom}(U, V)$. Using that $\mathrm{Hom}_G(U, V) = \mathrm{Hom}(U, V)^G$ (see (3) of Exercise 3.1), we get

$$\dim \mathrm{Hom}_G(U, V) = \mathrm{tr}_{\mathrm{Hom}(U,V)}(\epsilon) = |G|^{-1} \sum_{g\in G} \chi_{\mathrm{Hom}(U,V)}(g) = |G|^{-1} \sum_{g\in G} \chi_V(g)\overline{\chi}_U(g) =$$
$$= (\chi_V, \chi_U).$$

□

Corollary 3.10. *The characters of irreducible representations form an orthonormal basis of the hermitian vector space $\mathbb{C}[G]^G$.*

Proof. The number of (isomorphism classes of) irreducible representations coincides with the number of conjugacy classes by (2) of Corollary 3.3. The latter is the same as $\dim \mathbb{C}[G]^G$. The claim that the characters of irreducible representations form an orthonormal collection follows from the Schur lemma, Theorem 2.8, combined with Proposition 3.9. □

3.4. **Positive characteristic case.** In the case when $\mathbb{F}$ has positive characteristic, say p, results of Section 3.2 may fail. They are still true when $|G|$ is coprime to p but this case is boring.

Example 3.11. Let $\operatorname{char}\mathbb{F} = 3$. Consider the group S_3. In characteristic 0 (and in characteristic bigger than 3) it has three irreducible representations, the one-dimensional representations $\mathrm{triv}_3, \mathrm{sgn}_3$ and the two-dimensional representation refl_3. It turns out that over $\mathbb{F}$ there are only two irreducible representations, triv_3 and sgn_3, while refl_3 has a subrepresentation isomorphic to triv_3 with quotient isomorphic to sgn_3. This is because $\{(x,x,x)|x \in \mathbb{F}\} \subset \{(x_1,x_2,x_3)|x_1+x_2+x_3=0\}$.

In general, the representation theory of finite groups in positive characteristic is much more complicated than in characteristic 0. This is a subject of great current interest.

3.5. **Induction for representations of groups.** Finally, let us discuss the induction for representations of finite groups (i.e., for their group algebras). Let $H \subset G$ be finite groups and $\mathbf{k}$ be a commutative ring.

We can view $\mathbf{k}G$ as a $\mathbf{k}G$-$\mathbf{k}H$-bimodule for the action of G on the left and the action of H on the right. So, for a $\mathbf{k}H$-module V, we have the induced G-module:

$$\mathbf{k}G \otimes_{\mathbf{k}H} U.$$

Similarly, we can view $\mathbf{k}G$ as a $\mathbf{k}H$-$\mathbf{k}G$-bimodule for the action of H on the left and the action of G on the right. This gives rise to the $\mathbf{k}G$-module $\operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, U)$.

Proposition 3.12. *We have a natural isomorphism* $\mathbf{k}G \otimes_{\mathbf{k}H} U \cong \operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, U)$.

Proof. Consider the $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H)$, where we consider the left H-action on $\mathbf{k}G$. Explicitly, it consists of all maps $\varphi : \mathbf{k}G \to \mathbf{k}H$ such that $\varphi(hg) = h\varphi(g)$ for all $h \in H, g \in G$. This $\mathbf{k}$-module upgrades to an $\mathbf{k}G$-$\mathbf{k}H$-bimodule by

$$[g\varphi](g') := \varphi(g'g), \quad [\varphi h](g') := \varphi(hg').$$

Note that $\mathbf{k}G$ is a free left $\mathbf{k}H$-module (with basis labelled by the orbits of the left action of H on G). We leave it as an exercise to check that the homomorphism $\operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H) \otimes_{\mathbf{k}H} U \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, U)$ given by $\alpha \otimes u \mapsto [x \mapsto \alpha(x)u]$ is an isomorphism of bimodules.

So to prove the proposition it is enough to established an isomorphism

$$\mathbf{k}G \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H)$$

of $\mathbf{k}G$-$\mathbf{k}H$-bimodules, where on the left we have the bimodule used to define $\mathbf{k}G \otimes_{\mathbf{k}H} \bullet$. We send $g \in G$ to $\varphi_g \in \operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H)$ defined on a basis element $g' \in G$ by

$$\varphi_g(g') = \begin{cases} g'g, \text{if } g'g \in H, \\ 0, \text{else}, \end{cases} \tag{3.4}$$

and extend it to $\mathbf{k}G$ by the $\mathbf{k}$-linearity. We need to show that this map is G-equivariant, H-equivariant, and is an isomorphism of $\mathbf{k}$-modules.

H-equivariance: $\varphi_{gh}(g')$ and $\varphi_g(g')h$ are zero unless $g'g \in H$. And if $g'g \in H$, then $\varphi_{gh}(g') = g'gh = \varphi_g(g')h = [\varphi_g h](g')$.

G-equivariance. For $g_1 \in G$ we compute $\varphi_{g_1g}(g')$ and $[g_1\varphi_g](g')$. We have $\varphi_{g_1g}(g') = g'g_1g$ if $g'g_1g \in H$ and zero else. Next, we have $[g_1\varphi_g](g') = \varphi_g(g'g_1) = g'g_1g$ if this is an element of H and zero else. This establishes the G-equivariance.

Isomorphism of $\mathbf{k}$*-modules.* The map is $\mathbf{k}$-linear by construction. To show it is an isomorphism we need to check that the maps φ_g defined by (3.4) form a basis in $\operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H)$. Pick representatives $g_1, \ldots, g_k$ of the left H-orbits in G. We can identify $\operatorname{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H)$ with $(\mathbf{k}H)^{\oplus k}$ via $\varphi \mapsto (\varphi(g_i))_{i=1}^k$. For each $g \in G$ there is a unique index $i \in \{1, \ldots, k\}$ and a unique element $h \in H$ such that $hg_i = g^{-1}$. Then a direct check shows that under the isomorphism $\mathrm{Hom}_{\mathbf{k}H}(\mathbf{k}G, \mathbf{k}H) \xrightarrow{\sim} (\mathbf{k}H)^{\oplus k}$ the element φ_g is sent to h in the ith summand. The claim about an isomorphism follows. □

The G-module $\mathbf{k}G \otimes_{\mathbf{k}H} U \cong \mathrm{Hom}_{\mathbf{k}H}(\mathbf{k}G, U)$ is called *induced* from U and is denoted by $\mathrm{Ind}_H^G U$. Applying special cases of the Tensor-Hom adjunction, (1.1) and (1.2), Section 1.6, we get the following claim known as the Frobenius reciprocity.

Corollary 3.13. *For representations U of H and V of G we have the following natural isomorphisms:*

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(U), V) \cong \mathrm{Hom}_H(U, V), \quad \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(U)) \cong \mathrm{Hom}_H(V, U).$$

To finish this section we want to outline another, more classical, realization of $\mathrm{Ind}_H^G U$, closely related to the coinduction realizations. Namely, consider the set of all maps $G \to U$, denote it by $\mathsf{Fun}(G, U)$. The $\mathbf{k}$-module structure on U induces a $\mathbf{k}$-module structure on $\mathsf{Fun}(G, U)$. The action of G on itself from the right gives rise to an action of G on $\mathsf{Fun}(G, U)$ turning the latter to a $\mathbf{k}G$-module. Define the subset $\mathsf{Fun}_H(G, U) \subset \mathsf{Fun}(G, U)$ consisting of all maps f satisfying $f(hg) = hf(g)$, i.e., equivariant for the action of H on G from the left. Note that $\mathsf{Fun}_H(G, U)$ is a $\mathbf{k}G$-submodule in $\mathsf{Fun}(G, U)$. It is naturally identified with $\mathrm{Hom}_{\mathbf{k}H}(\mathbf{k}G, U)$.