Introduction to the quantized enveloping algebra

References [Jantzen, Lectures on quantum groups, §4,5]
[Lusztig, Introduction to quantum groups]
[Etingof-Gelaki-Nykshych-Ostrik, Tensor categories]

Part I: the algebra $U_q(\mathfrak{g})$

Notation $\mathfrak{g}$ complex semisimple Lie algebra
$\cup$
$\mathfrak{h}$ Cartan subalgebra

Φ root system, W Weyl group
$\cup$
Π set of simple roots

$(-,-)$ W-invariant scalar product on $\mathbb{Q}\otimes\mathbb{Z}\Phi$ defined on each irreducible component by $(\alpha,\alpha)=2$ for any short root.
(so we can have $(\beta,\beta)=2,4,6$, for $\beta\in\Phi$)

$$\Rightarrow \langle\lambda,\alpha^\vee\rangle = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$$

$\Lambda = \{\lambda\in\mathbb{Q}\otimes\mathbb{Z}\Phi \mid \langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}, \forall\alpha\in\Phi\}$ wt lattice

$$\forall\lambda\in\Lambda,\quad (\lambda,\alpha) = \langle\lambda,\alpha^\vee\rangle\frac{(\alpha,\alpha)}{2}\in\mathbb{Z}$$

$\{\varpi_\alpha\}_{\alpha\in\Pi}$ fundamental weights.

Let k be a field, $\operatorname{char} k \neq 2$

(and $\neq 3$ if Φ has irred. comp. of type G_2)

Fix $q \in k^\times$ s.t. $q^{(\alpha,\alpha)} \neq 1$ $\forall \alpha \in \Phi$

Set $d_\alpha := \frac{(\alpha,\alpha)}{2}$, $q_\alpha = q^{d_\alpha}$ $\forall \alpha \in \Pi$

So the condition says $q_\alpha^2 \neq 1$

Quantum numbers and Gaussian binomial coefficients

$a \in \mathbb{Z}$ $\quad [a] := \dfrac{v^a - v^{-a}}{v - v^{-1}} \in \mathbb{Z}[v, v^{-1}]$

$n \in \mathbb{N}$ $\quad \begin{bmatrix} a \\ n \end{bmatrix} := \dfrac{[a][a-1]\cdots[a-n+1]}{[n]\cdots[1]} \in \mathbb{Z}[v, v^{-1}]$

Let $\varphi : \mathbb{Z}[v, v^{-1}] \xrightarrow{v \mapsto q} k$, $\quad [a]_{v=q} := \varphi([a])$

$\begin{bmatrix} a \\ n \end{bmatrix}_{v=q} := \varphi\left(\begin{bmatrix} a \\ n \end{bmatrix}\right)$

We denote $[a]_\alpha$ and $\begin{bmatrix} a \\ n \end{bmatrix}_\alpha$, $[a]_{v=q_\alpha}$ and $\begin{bmatrix} a \\ n \end{bmatrix}_{v=q_\alpha}$ respectively.

Definition of $U_q(\mathfrak{g})$

$U_q(\mathfrak{g})$ is the associative k-algebra w/ generators

$$E_\alpha,\ F_\alpha,\ K_\alpha,\ K_\alpha^{-1} \qquad (\alpha \in \Pi)$$

and relations

1) $K_\alpha K_\alpha^{-1} = 1 = K_\alpha^{-1} K_\alpha\ , \qquad K_\alpha K_\beta = K_\beta K_\alpha$

2) $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$

3) $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$

4) $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \dfrac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$

and letting $r_{\alpha,\beta} = 1 - \langle \beta, \alpha^\vee \rangle$, for $\alpha \neq \beta$ in Π

5) $\displaystyle\sum_{i=0}^{r_{\alpha,\beta}} (-1)^i \begin{bmatrix} r_{\alpha,\beta} \\ i \end{bmatrix}_\alpha E_\alpha^i E_\beta E_\alpha^{r_{\alpha,\beta}-i} = 0$

6) $\displaystyle\sum_{i=0}^{r_{\alpha,\beta}} (-1)^i \begin{bmatrix} r_{\alpha,\beta} \\ i \end{bmatrix}_\alpha F_\alpha^i F_\beta F_\alpha^{r_{\alpha,\beta}-i} = 0$

eg. for $\langle \beta, \alpha^\vee \rangle = 0 = \langle \alpha, \beta^\vee \rangle$ we get $[E_\alpha, E_\beta] = 0$
$[F_\alpha, F_\beta] = 0$

for $\langle \beta, \alpha^\vee \rangle = -1$ we get $E_\alpha^2 E_\beta - [2]_\alpha E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 = 0$
$F_\alpha^2 F_\beta - [2]_\alpha F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 = 0$

The subalgebra U^0

i) $U^0 := \langle K_\alpha^{\pm 1} \rangle_{\alpha \in \Pi}$ is commutative

denote $K_\gamma := \prod_{\alpha \in \Pi} K_\alpha^{m_\alpha}$ where $\gamma = \sum_{\alpha \in \Pi} m_\alpha \alpha$

then $K_\gamma K_\delta = K_{\gamma+\delta}$ $\forall \gamma, \delta \in \mathbb{Z}\Phi$

ii) By relations (3) and (4) we get

$$K_\gamma E_\alpha K_\gamma^{-1} = q^{(\gamma,\alpha)} E_\alpha$$

$$K_\gamma F_\alpha K_\gamma^{-1} = q^{-(\gamma,\alpha)} F_\alpha$$

iii) One could extend the Cartan part U^0 by considering any subgroups $\Gamma \subset \mathbb{Q} \otimes \mathbb{Z}\Phi$ such that $(\gamma, \alpha) \in \mathbb{Z}$ $\forall \gamma \in \Gamma, \alpha \in \Pi$ (e.g. $\Gamma = \Lambda$) and hence replacing the generators K_α's with K_γ $(\gamma \in \Gamma)$ and relation (1) by those in rmks (i) and (ii)

Hopf algebra structure on $U_q(\mathfrak{g})$

We denote $U := U_q(\mathfrak{g})$

<u>Lemma</u> There exist unique morphisms of k-algebras

$$\Delta : U \longrightarrow U \otimes U$$

$$S : U \longrightarrow U^{opp}$$

$$\varepsilon : U \longrightarrow k$$

such that, $\forall \alpha \in \Pi$

$$\Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$$

$$\Delta(F_\alpha) = 1 \otimes F_\alpha + F_\alpha \otimes K_\alpha^{-1}$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha \qquad \Delta(K_\alpha^{-1}) = K_\alpha^{-1} \otimes K_\alpha^{-1}$$

$$S(E_\alpha) = -K_\alpha^{-1} E_\alpha \qquad \varepsilon(E_\alpha) = 0$$

$$S(F_\alpha) = -F_\alpha K_\alpha \qquad \varepsilon(F_\alpha) = 0$$

$$S(K_\alpha) = K_\alpha^{-1} \qquad \varepsilon(K_\alpha) = 1$$

$$S(K_\alpha^{-1}) = K_\alpha \qquad \varepsilon(K_\alpha^{-1}) = 1$$

<u>Proposition</u> (Δ, ε, S) defines a Hopf algebra structure on U

Pf One can check on generators that

i) Δ is <u>coassociative</u>, i.e.

$$(\Delta \otimes 1)\circ \Delta = (1 \otimes \Delta)\circ \Delta$$

ii) ε satisfies <u>counit</u> axiom, i.e.

$$\begin{tikzcd} U \arrow[r, "\Delta"] \arrow[dr, "\mathrm{can}"'] & U \otimes U \arrow[d, "\varepsilon \otimes 1"] \\ & k \otimes U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\Delta"] \arrow[dr, "\mathrm{can}"'] & U \otimes U \arrow[d, "1 \otimes \varepsilon"] \\ & U \otimes k \end{tikzcd}$$

commute

iii) S satisfies the <u>antipode</u> axiom i.e.

$$\begin{tikzcd} U \arrow[r, "\Delta"] \arrow[d, "\iota\circ\varepsilon"'] & U \otimes U \arrow[d, "S \otimes 1"] \\ U & U \otimes U \arrow[l, "m"] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\Delta"] \arrow[d, "\iota\circ\varepsilon"'] & U \otimes U \arrow[d, "1 \otimes S"] \\ U & U \otimes U \arrow[l, "m"] \end{tikzcd}$$

commute, where m is the multiplication and $\iota : k \longrightarrow U$ is the morphism

$$x \longmapsto x \cdot 1_U$$

so $\iota\circ\varepsilon$ sends u to $\varepsilon(u)\cdot 1_U$

Proof of Lemma (sketch)

We have to check relations:

Relation 1) easy

Relation 2) and 3) One can use a natural $\mathbb{Z}\Phi$-grading on U, given by

$\deg(E_\alpha) = \alpha$, $\deg(F_\alpha) = -\alpha$, $\deg(K_\alpha) = 0$

and check that

$$(*) \quad K_\gamma u K_\gamma^{-1} = q^{(\lambda,\gamma)} u \quad \text{if } \deg(u) = \lambda \in \mathbb{Z}\Phi$$

Then one can check that the definition of Δ, ε and S preserve the grading (on $U \otimes U$ we have an induced grading and a relation analogous to $(*)$)

Relation 4) $[\Delta(E_\alpha), \Delta(F_\beta)] = [E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha, 1 \otimes F_\beta + F_\beta \otimes K_\beta^{-1}]$

$$= \cancel{[E_\alpha \otimes 1, 1 \otimes F_\beta]}^{\,0} + [E_\alpha, F_\beta] \otimes K_\beta^{-1} + K_\alpha \otimes [E_\alpha, F_\beta] + \cancel{[K_\alpha \otimes E_\alpha, F_\beta \otimes K_\beta^{-1}]}^{\,0}$$

$$= \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \otimes K_\beta^{-1} + K_\alpha \otimes \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$ (using rel. 2, 3)

$$= \delta_{\alpha,\beta} \frac{K_\alpha \otimes K_\alpha - K_\alpha^{-1} \otimes K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} = \Delta\left(\delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}\right)$$

Rmk If $\rho := \frac{1}{2}\sum_{\alpha \in \Phi} \alpha = \sum_{\alpha \in \Pi} \varpi_\alpha$, we have

$$(2\rho, \alpha) = 2(\rho, \alpha) = \langle \rho, \alpha^\vee \rangle (\alpha, \alpha) = (\alpha, \alpha)$$

So $K_{2\rho}^{-1} E_\alpha K_{2\rho} = q^{-(2\rho, \alpha)} E_\alpha = q_\alpha^{-2} E_\alpha$

$K_{2\rho}^{-1} F_\alpha K_{2\rho} = q^{+(2\rho, \alpha)} F_\alpha = q_\alpha^{2} F_\alpha$

Hence one can check that

$$S^2(u) = K_{2\rho}^{-1} u K_{2\rho} \quad \forall u \in U.$$

$$[S(E_\alpha), S(F_\beta)] = [-K_\alpha^{-1}E_\alpha, -F_\beta K_\beta] =$$

again using rel. 2,3

$$= K_\alpha^{-1}[E_\alpha, F_\beta]K_\beta = \delta_{\alpha,\beta}\, K_\alpha^{-1}\, \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}\, K_\beta$$

$$= \delta_{\alpha,\beta}\, \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} = -S\left(\delta_{\alpha,\beta}\, \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}\right)$$

$$[\varepsilon(E_\alpha), \varepsilon(F_\beta)] = 0 = \varepsilon\left(\delta_{\alpha,\beta}\, \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}\right)$$

Relations 5) and 6) We will treat (5) in the A_2-case to give an idea. We have $\langle \beta, \alpha^\vee \rangle = -1$ so $n_{\alpha,\beta} = 2$:

$$\sum_{i=0}^{2} (-1)^i \begin{bmatrix} 2 \\ i \end{bmatrix}_\alpha E_\alpha^i E_\beta E_\alpha^{2-i} = 0$$

We have:

$$\Delta(E_\alpha)^2 \Delta(E_\beta) - [2]_\alpha \Delta(E_\alpha)\Delta(E_\beta)\Delta(E_\alpha) + \Delta(E_\beta)\Delta(E_\alpha)^2$$

$$= (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)^2 (E_\beta \otimes 1 + K_\beta \otimes E_\beta) +$$

$$- [2]_\alpha (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)(E_\beta \otimes 1 + K_\beta \otimes E_\beta)(E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)$$

$$+ (E_\beta \otimes 1 + K_\beta \otimes E_\beta)(E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)^2 =$$

$$= (E_\alpha^2 \otimes 1 + (1+q_\alpha^2) E_\alpha K_\alpha \otimes E_\alpha + K_\alpha^2 \otimes E_\alpha^2)(E_\beta \otimes 1 + K_\beta \otimes E_\beta)$$

$$- [2]_\alpha (E_\alpha E_\beta E_\alpha \otimes 1 + E_\alpha E_\beta K_\alpha \otimes E_\alpha + E_\alpha K_\beta E_\alpha \otimes E_\beta + E_\alpha K_\beta K_\alpha \otimes E_\beta E_\alpha$$

$$+ K_\alpha E_\beta E_\alpha \otimes E_\alpha + K_\alpha E_\beta K_\alpha \otimes E_\alpha^2 + K_\alpha K_\beta E_\alpha \otimes E_\alpha E_\beta + K_\alpha K_\beta K_\alpha \otimes E_\alpha E_\beta E_\alpha)$$

$$+ (E_\beta \otimes 1 + K_\beta \otimes E_\beta)(E_\alpha^2 \otimes 1 + (1+q_\alpha^2) E_\alpha K_\alpha \otimes E_\alpha + K_\alpha^2 \otimes E_\alpha^2)$$

$$= E_\alpha^2 E_\beta \otimes 1 + E_\alpha^2 K_\beta \otimes E_\beta + (1+q_\alpha^2)\, q^{(\alpha,\beta)} E_\alpha E_\beta K_\alpha \otimes E_\alpha$$

$$+ (1+q_\alpha^2)\, E_\alpha K_\alpha K_\beta \otimes E_\alpha E_\beta + q^{2(\alpha,\beta)} E_\beta K_\alpha^2 \otimes E_\alpha^2 +$$

$$+ K_\alpha^2 K_\beta \otimes E_\alpha^2 E_\beta$$

$$- [2]_\alpha E_\alpha E_\beta E_\alpha \otimes 1 - [2]_\alpha E_\alpha E_\beta K_\alpha \otimes E_\alpha - q^{(\alpha,\beta)} [2]_\alpha E_\alpha^2 K_\beta \otimes E_\beta$$

$$- [2]_\alpha E_\alpha K_\alpha K_\beta \otimes E_\beta E_\alpha - q^{(\alpha,\alpha+\beta)} [2]_\alpha E_\beta E_\alpha K_\alpha \otimes E_\alpha +$$

$$- q^{(\alpha,\beta)} [2]_\alpha E_\beta K_\alpha^2 \otimes E_\alpha^2 - q^{(\alpha,\alpha+\beta)} [2]_\alpha E_\alpha K_\alpha K_\beta \otimes E_\alpha E_\beta +$$

$$- [2]\, K_\alpha^2 K_\beta \otimes E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 \otimes 1 + (1+q_\alpha^2)\, E_\beta E_\alpha K_\alpha \otimes E_\alpha$$

$$+ E_\beta K_\alpha^2 \otimes E_\alpha^2 + q^{2(\alpha,\beta)} E_\alpha^2 K_\beta \otimes E_\beta + (1+q_\alpha^2)\, q^{(\alpha,\beta)} E_\alpha K_\alpha K_\beta \otimes E_\beta E_\alpha$$

$$+ K_\alpha^2 K_\beta \otimes E_\beta E_\alpha^2 =$$

$$= \left(E_\alpha^2 E_\beta - [2]_\alpha E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2\right) \otimes 1 +$$

$$+ E_\alpha^2 K_\beta \otimes E_\beta \left(1 - q^{(\alpha,\beta)} [2]_\alpha + q^{2(\alpha,\beta)}\right) +$$

$$+ E_\alpha E_\beta K_\alpha \otimes E_\alpha \left((1+q_\alpha^2)\, q^{(\alpha,\beta)} - [2]_\alpha\right) +$$

$$+ E_\alpha K_\alpha K_\beta \otimes E_\alpha E_\beta \left((1+q_\alpha^2) - q^{(\alpha,\alpha+\beta)} [2]_\alpha\right) +$$

$$+ E_\beta K_\alpha^2 \otimes E_\alpha^2 \left(q^{2(\alpha,\beta)} - q^{(\alpha,\beta)} [2]_\alpha + 1\right) +$$

$$+ K_\alpha^2 K_\beta \otimes \left(E_\alpha^2 E_\beta - [2]_\alpha E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2\right) +$$

$$+ E_\alpha K_\alpha K_\beta \otimes E_\beta E_\alpha \left((1+q_\alpha^2)\, q^{(\alpha,\beta)} - [2]_\alpha\right) +$$

$$+ E_\beta E_\alpha K_\alpha \otimes E_\alpha \left(1 + q_\alpha^2 - q^{(\alpha,\alpha+\beta)} [2]_\alpha\right) =$$

$$= 0$$

$$S(E_\beta)S(E_\alpha)^2 - [2]_\alpha S(E_\alpha)S(E_\beta)S(E_\alpha) + S(E_\alpha)^2 S(E_\beta)$$

$$= (-K_\beta^{-1}E_\beta)(-K_\alpha^{-1}E_\alpha)^2 - [2]_\alpha(-K_\alpha^{-1}E_\alpha)(-K_\beta^{-1}E_\beta)(-K_\alpha^{-1}E_\alpha)$$

$$+ (-K_\alpha^{-1}E_\alpha)^2(-K_\beta^{-1}E_\beta) =$$

$$= -q^{+(\alpha,\beta)+(\alpha,\alpha+\beta)} K_\alpha^{-2}K_\beta^{-1}E_\beta E_\alpha^2 +$$

$$+[2]_\alpha q^{(\alpha,\beta)+(\alpha,\alpha+\beta)} K_\alpha^{-2}K_\beta^{-1}E_\alpha E_\beta E_\alpha +$$

$$- q^{(\alpha,\beta)+(\alpha,\alpha+\beta)} K_\alpha^{-2}K_\beta^{-1}E_\alpha^2 E_\beta = 0$$

and for ε it is obvious

$\square$

The adjoint representation

Given any Hopf algebra $(A, \Delta, \varepsilon, S)$ one can give A the structure of a $(A \otimes A)$-module via

$$(a_1 \otimes a_2) \cdot b := a_1 \, b \, S(a_2)$$

This gives, via Δ, an A-module structure called the adjoint representation ad

If $\Delta(a) = \sum_i a_i \otimes a_i'$, then

$$\mathrm{ad}(a)\, b = \sum_i a_i \, b \, S(a_i')$$

If we apply this to U, we get

$$\mathrm{ad}(E_\alpha)\, u = E_\alpha u - K_\alpha u K_\alpha^{-1} E_\alpha$$
$$\mathrm{ad}(F_\alpha)\, u = (u F_\alpha - F_\alpha u) K_\alpha$$
$$\mathrm{ad}(K_\alpha)\, u = K_\alpha u K_\alpha^{-1}$$

Rmk Relations (5) and (6) can be rewritten

$$\mathrm{ad}(E_\alpha^{r})\, E_\beta = 0 \qquad \mathrm{ad}(F_\alpha^{r})(F_\beta K_\beta) = 0$$

where $r = r_{\alpha\beta} = 1 - \langle \beta, \alpha^\vee \rangle$

Triangular decomposition

Recall $U^0 := \langle K_\alpha^{\pm 1} \rangle_{\alpha \in \Pi}$, and set

$$U^+ := \langle E_\alpha \rangle_{\alpha \in \Pi} \subset U$$
$$U^- := \langle F_\alpha \rangle_{\alpha \in \Pi} \subset U$$

then we have

<u>Theorem</u> The multiplication map

$$U^- \otimes U^0 \otimes U^+ \xrightarrow{m} U$$

is an isomorphism (of v.s.)

<u>Rmk</u> One can define an automorphism

$\omega : U \xrightarrow{\sim} U$ given by

$\omega(E_\alpha) = F_\alpha$, $\omega(F_\alpha) = E_\alpha$, $\omega(K_\alpha) = K_\alpha^{-1}$

then clearly $\omega(U^-) = U^+$, $\omega(U^+) = U^-$

So we get also the opposite decomposition

$$U^+ \otimes U^0 \otimes U^- \xrightarrow{\sim} U$$

Pf (sketch)

Step 1 Consider $\tilde{U}$, the algebra with same generators as U but only relations (1) to (4)

Let $\tilde{U}^\circ = \langle K_\alpha^{\pm 1} \rangle$, $\tilde{U}^+ = \langle E_\alpha \rangle$, $\tilde{U}^- = \langle F_\alpha \rangle$ inside $\tilde{U}$

Here one can show that monomials of the form

$$F_{\alpha_1} \cdots F_{\alpha_k} K_\mu E_{\beta_1} \cdots E_{\beta_n} \qquad (\alpha_i, \beta_j \in \Pi,\ \mu \in \mathbb{Z}\phi)$$

form a basis of $\tilde{U}$

Step 2 Let $I := \ker(\tilde{U} \twoheadrightarrow U)$, so I is the two-sided ideal generated by all RHS's of relations (5)-(6)

Now let $I^+ \subset \tilde{U}^+$ (resp. $I^- \subset \tilde{U}^-$) be the two-sided ideal in $\tilde{U}^+$ (resp. $\tilde{U}^-$) generated by all RHS's of relation (5) (resp. (6)).

Then one can show

1) $\tilde{U}^+ / I^+ \cong U^+$
2) $\tilde{U}^- / I^- \cong U^-$
3) The image by the multiplication map of
$$\tilde{U}^- \otimes \tilde{U}^\circ \otimes I^+ + I^- \otimes \tilde{U}^\circ \otimes \tilde{U}^+ \quad \text{is } I$$

So, observing that $\tilde{U}^\circ \cong U^\circ$, we get

$$U = \tilde{U}/I \xleftarrow{\sim} \frac{\tilde{U}^- \otimes \tilde{U}^\circ \otimes \tilde{U}^+}{(\tilde{U}^- \otimes \tilde{U}^\circ \otimes I^+ + I^- \otimes \tilde{U}^\circ \otimes \tilde{U}^+)} \cong \frac{\tilde{U}^-}{I^-} \otimes \tilde{U}^\circ \otimes \frac{\tilde{U}^+}{I^+}$$

$$\cong U^- \otimes U^\circ \otimes U^+$$

$\square$