§. Linkage Principle

• Set up.

• Γ: field of char 0

• $f: \mathcal{B} \to \Gamma$ specialization of $\mathcal{B}$ into Γ which takes v into a primitive l-th root of 1.

• $U_\Gamma = U_{\mathcal{B}} \otimes_{\mathcal{B}} \Gamma$.

Recall We define the affine Weyl group in my first talk. Now, let l define $W_l^a := W \ltimes \Lambda_r$, and the "$l$-rescaled dot action" on Λ by

$$w \cdot_l \lambda = w \cdot \lambda, \quad t_v \cdot_l \lambda := \lambda + l v, \quad \forall\, w \in W,\ v \in \Lambda_r.$$

So, W_l^a is generated by reflections $s_{\alpha,r}: \Lambda \to \Lambda$, $\alpha \in \Phi^+$, $r \in \mathbb{Z}$ where

$$s_{\alpha,r}(\lambda) := s_\alpha(\lambda) + r l \alpha, \quad \forall\, \lambda \in \Lambda.$$

Example Let $G = SL_2$. $\Lambda = \mathbb{Z}\chi_1$, $\Lambda_r = 2\mathbb{Z}\chi_1$, where $\chi_1 : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t$.

$W = \{1, s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$. For any $n\chi_1 \in \Lambda$, $s(n\chi_1) = -n\chi_1$.

For any $\lambda = n\chi_1 \in \Lambda$ and $v = 2m\chi_1 \in \Lambda_r$,

$$\begin{aligned}(s, t_{2m\chi_1}) \cdot_l n\chi_1 &= s(n\chi_1 + 2ml\chi_1 + \chi_1) - \chi_1 \\ &= -(n + 2ml + 2)\chi_1\end{aligned}$$

$$\Rightarrow \quad n\chi_1 \geq (s, t_{2m\chi_1}) \cdot_l n\chi_1 \iff n \geq -ml-1.$$

$$(1, t_{2m\chi_1}) \cdot_l n\chi_1 = n\chi_1 + 2ml\chi_1 \quad \Rightarrow \quad n\chi_1 \geq (1, t_{2m\chi_1}) \cdot_l n\chi_1 \iff m \leq 0$$

$(s, t_{2m'x_1}) \cdot_\ell (-n-2m\ell-2)x_1$

$= s((-n-2m\ell-2)x_1 + 2m'\ell x_1 + x_1) - x_1$

$= (n+2m\ell-2m'\ell) x_1$

$\Rightarrow \quad n x_1 \leq (s, t_{2m'x_1}) \cdot_\ell (-n-2m\ell-2)x_1 \iff m-m' \geq 0$

So, the above computation allows us to conclude that $(\ell-1)x_1$ is minimal in its $\hat{W_\ell}$ orbit.

Remark In general, $(\ell-1)\rho$ is the minimal element in its $\hat{W_\ell}$ orbit.

As in the case of U_ϵ, we have the following theorem.

· Thm. For any $\lambda \in \Lambda^+$, $W_\Gamma(\lambda)$ has a unique simple quotient $L_\Gamma(\lambda)$ whose highest weight is λ. Moreover, any finite-dim simple U_Γ-module (of type 1) is isom. to $L_\Gamma(\lambda)$ for some $\lambda \in \Lambda^+$.

· Remark (i) We define the dual Weyl module $M_\Gamma(\lambda) := W_\Gamma(-\lambda^*)^*$. Then above thm may be restated as .... has a unique submodule $L_\Gamma(\lambda)$, ...

(ii) We define the dual Weyl module as the dual of Weyl modules. In fact we can define it using an induction functor as we have seen in my first talk $\mathrm{Ind}_{U_\Gamma^{\leq 0}}^{U_\Gamma} : \mathcal{C}_\Gamma^{\leq 0} \to \mathcal{C}_\Gamma$, $\mathcal{C}_\Gamma^{\leq 0}$ (resp.) $\mathcal{C}_\Gamma$ is the cat. of integrable $U_\Gamma^{\leq 0}$ (resp.) U_Γ modules. Then $M_\Gamma(\lambda) \simeq \mathrm{Ind}_{U_\Gamma^{\leq 0}}^{U_\Gamma} T_\lambda$.

(iii) The induction functor has the desired props

- (Frob. Reci) $\forall M \in \mathcal{C}_P^{\le 0}$, $N \in \mathcal{C}_P$, then

$$\mathrm{Hom}_{U_P}(N, \mathrm{Ind}_{U_P}^{U_P^{\le 0}} M) \simeq \mathrm{Hom}_{U_P^{\le 0}}(N, M)$$

- (Tensor Identity) V : P-module, $M \in \mathcal{C}_P^{\le 0}$. Then there is a U_P-module isom.

$$V \otimes \mathrm{Ind}_{U_P^{\le 0}}^{U_P} M \simeq \mathrm{Ind}_{U_P^{\le 0}}^{U_P}(V \otimes M).$$

- Higher derived functors

- Kempf's Vanishing Thm

For a more detailed discussion, see [APW].

- Thm (Weak Linkage Principle). Let $\lambda_1, \lambda_2 \in \Lambda^+$. if $\mathrm{Ext}^1(L_P(\lambda_1), L_P(\lambda_2)) \neq 0$, then $\lambda_1 \in W_e^a \cdot_e \lambda_2$.

§. Steinberg Representations and Proj. Objects.

Again, we assume that char $\mathbb{T}>0$ in this section.

- Lemma. The Steinberg rep. $W_{p}((\ell-1)\rho)$ is simple.

Proof. It follows from the weak Linkage principle and and the fact that $(\ell-1)\rho$ is minimal in its W_ℓ^a orbit.

- Prop. For any $\lambda\in\Lambda^+$, $W_p((\ell-1)\rho+\ell\lambda)$ is simple.

Proof. By definition, $W_p((\ell-1)\rho+\ell\lambda)\twoheadrightarrow L_p((\ell-1)\rho+\ell\lambda)$. We compare their dimensions.

By the Weyl dimension formula,

$$\dim W_p((\ell-1)\rho+\ell\lambda)=\prod_{\alpha\in\Phi^+}\frac{\langle(\ell-1)\rho+\ell\lambda+\rho,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle}=\ell^{|\Phi^+|}\prod_{\alpha\in\Phi^+}\frac{\langle\lambda+\rho,\alpha^\vee\rangle}{\langle\rho,\alpha^\vee\rangle}$$

$$=\ell^{|\Phi^+|}\dim W_p(\lambda)$$

By Lusztig's Tensor Product Thm,

$$L_p((\ell-1)\rho+\ell\lambda)\simeq L_p((\ell-1)\rho)\otimes \mathrm{Fr}^* L(\lambda).$$

Since $L_p((\ell-1)\rho)\simeq W_p((\ell-1)\rho)$, $\dim(L_p((\ell-1)\rho+\ell\lambda))=\dim W_p((\ell-1)\rho)\dim L(\lambda)$
$=\ell^{|\Phi^+|}\dim L(\lambda)$.

Note that as $U(\mathfrak{g})$-modules, $W_p(\lambda)$ and $L(\lambda)$ ~~are~~ have the same set of generators and relations, then $\dim W_p(\lambda)=\dim L(\lambda)$

$\square$

• Notations.

$\mathcal{F}_\Gamma$: the cat. of all finite-dim U_Γ-module of type 1.

Thm. St is a proj. object in $\mathcal{F}_\Gamma$.

Proof It suffices to show that
$$\mathrm{Ext}^1_{U_\Gamma}(St, L(\lambda)) = 0 \ \forall \ \lambda \in \Lambda^+ \ (*)$$

Weak Linkage principle $\Rightarrow$ suffices to show this for $\lambda \in W^a_{\ell} \cdot (\ell-1)\rho$. Thus, we let $\lambda = (\ell-1)\rho + \ell\mu$ for some $\mu \in \Lambda^+$. We have shown that $W_\Gamma((\ell-1)\rho + \ell\lambda) \simeq L_\Gamma((\ell-1)\rho + \ell(\lambda))$. Thus, it suffices to prove that
$$\mathrm{Ext}^1_{U_\Gamma}(St, W_\Gamma(\lambda)) = 0.$$

This follows from the following lemma

Lemma. $\forall \lambda_1, \lambda_2 \in \Lambda^+$, $\mathrm{Ext}^1_{U_\Gamma}(W_\Gamma(\lambda_1), M_\Gamma(\lambda_2)) = 0$

The above lemma essentially follows from the universal prop. of quantum Weyl modules

$\square$

Remark. $St \simeq St^* \Rightarrow St$ is also proj.

Lemma. For any $\lambda \in \Lambda^+$, $\exists$ an embedding $L_\Gamma(\lambda) \hookrightarrow St \otimes E$ for some $E \in \mathcal{F}_\Gamma$.

Proof Lusztig's Tensor Product Thm $\Rightarrow$ may assume that λ is a restricted weight. Then, $\mu := (\ell-1)\rho - \lambda \in \Lambda^+$, and the natural $U_\Gamma^{\leq 0}$-homomorphism $L(\lambda) \otimes L(\mu) \to \Gamma_{(\ell-1)\rho}$ gives rise to a U_Γ-homomorphism
$$L(\lambda) \hookrightarrow L(\mu)^* \otimes St.$$

$\square$

• **Thm.** (i) $\mathcal{F}_\Gamma$ has enough injectives. Moreover, any injective object is a direct summand of $St \otimes E$, for some $E \in \mathcal{F}_\Gamma$.

(ii) Injectives $\Leftrightarrow$ Projectives in $\mathcal{F}_\Gamma$.

Proof (i) Recall that $\forall M \in \mathcal{F}_\Gamma$, $soc(M) :=$ sum of its irred. submodules. Then $soc(M) = \bigoplus_{\mu_i \in \Lambda^+} L_\Gamma(\mu_i)$. Then $soc(M) \hookrightarrow St \otimes E$ for some $E \in \mathcal{F}_\Gamma$, and $St \otimes E$ is ~~proj~~ inj. So, we get $M \hookrightarrow St \otimes E$

(ii) Let M be an inj. object. Then $M \hookrightarrow St \otimes E$ as a direct summand since $St \otimes E$ is proj, then M is proj. The converse is obvious.