Prop 4 (a) Any Weyl mod V_λ^κ has a unique irred quotient (denoted by L_λ^κ).

(b) $(\lambda \neq \mu) \Rightarrow (L_\lambda^\kappa \not\cong L_\mu^\kappa)$.

(proof): • By coro 3, $\mathfrak{g} \curvearrowright \rho_\kappa(\lambda)(V_\lambda^\kappa)$ is irred.

so for any proper submod M of V_λ^κ,

$$M \cap {}_{\rho_\kappa(\lambda)} V_\lambda^\kappa = 0.$$

• Hence the maximal submodule is contained in $\bigoplus_{n \in \mathbb{N}_{\geq 1}} (\rho_\kappa(\lambda) - n)(V_\kappa^\lambda)$.

• So V_λ^κ has a unique irred quot. (proving (a)).

- Suppose $\lambda \neq \mu$. but $L_\lambda^\kappa \simeq L_\mu^\kappa$.

- Then

$$\begin{tikzcd} V_\lambda^\kappa \arrow[r, two heads] \arrow[rr, bend right, two heads] & L_\lambda^\kappa \arrow[r, phantom, "\simeq"] & L_\mu^\kappa \end{tikzcd} \quad \cdots \quad \begin{matrix} p_\kappa(\lambda) \\ \parallel \\ p_\kappa(\mu) \end{matrix}$$

-

$$\begin{tikzcd} p_\kappa(\lambda)(V_\lambda^\kappa) \arrow[r, "\sim"] \arrow[d, phantom, "\text{cor3}\ \wr|"] & p_\kappa(\lambda)(L_\lambda^\kappa) \arrow[r, "\sim"] & p_\kappa(\lambda)(L_\mu^\kappa) \arrow[d, phantom, "\wr|\ \text{coro 3.}"] \\ V_\lambda & & V_\mu \end{tikzcd}$$

$*$.

#

Coro 4.1 $\dim \mathrm{Hom}_{\hat{\mathfrak{g}}_k}(V_\lambda^k, L_\mu^k) = \delta_{\lambda\mu}$.

(proof) verbal.

Prop 5. (universal property of Weyl mods).

Let V_λ^k be a Weyl module, and N a $\hat{\mathfrak{g}}_k$-module.

Then $\mathrm{Hom}_{\hat{\mathfrak{g}}_k}(V_\lambda^k, N) \simeq \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, N(1))$.

(recall) $\{n \in N \mid t\mathfrak{g}\, n = 0\}$

Proof • By reciprocity, we have an equiv of vect space

$$\mathrm{Hom}_{\hat{\mathfrak{g}}_k}(V_\lambda^k, N) \simeq \mathrm{Hom}_{\mathfrak{g}[[t]]\oplus \mathbb{C}c}(V_\lambda, N)$$

- $N(1) \hookrightarrow N$ induces

$$\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, N(1)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}c}(V_\lambda, N).$$

$$\underset{\phi}{\cup}$$

- The latter map is onto, ...

$\because$ $\forall \phi \in \mathrm{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}c}(V_\lambda, N)$ $\forall v \in V_\lambda$

$$\phi(v) \in N(1).$$

#

<u>Corollary 5.</u> $\underline{V_\lambda \simeq L_\lambda^\kappa(1)}$ as $\mathfrak{g}$-module (so as a $\mathfrak{g}[t]\oplus\mathbb{C}c$ mod)

<u>proof</u> $\mathrm{Hom}_{\mathfrak{g}}(V_\mu, L_\lambda^\kappa(1)) \underset{\text{props}}{\simeq} \mathrm{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}c}(V_\mu, L_\lambda^\kappa) \simeq \mathrm{Hom}_{\hat{\mathfrak{g}}_\kappa}(V_\mu^\kappa, L_\lambda^\kappa)$

$\dim = \delta_{\mu,\lambda}$.

. #

Prop 6 Any generalized Weyl module has finite length.

recall assumption!

$$\left(\frac{\kappa}{\kappa_{Killing}}+\frac{1}{2}\right)\notin \mathbb{Q}_{\geq 0}.$$

(proof) • Claim It suffices to prove for Weyl modules b/c Prop1.

take one: V_λ^{κ}.

• (Exercise) The following set is finite

$$S := \{\delta \in \Lambda^{+} \mid P_\kappa(\lambda) - P_\kappa(\delta) \in \mathbb{N}_{\geq 1}\}$$

Recall

$$P_\kappa(\lambda) = \frac{1}{2}\frac{k}{k-k_{crit}}(\lambda, \lambda+2\rho)$$

• Claim For any $\hat{\mathfrak{g}}_\kappa$-modules $M_2 \subsetneq M_1 \subsetneq V_\lambda^{\kappa}$, $\exists\, m \in M_1 \setminus M_2$ s.t. m is a generalized L_0-eigenvector w/ corr eigenvalue in S.

- With claim we're done, b/c

$$\text{length}(V_\lambda^\kappa) \le \sum_{\delta \in S} \underbrace{\dim(\text{gen. eig. space}(\delta))}_{\substack{\wedge \\ \infty}\ \text{by prop 2.}}.$$

- (proof of claim)
  - Let $\tilde{M} := M_1/M_2$. The set of eigenvalues of $L_0 \curvearrowright \tilde{M}$ is bounded above and discrete.
  - Let σ to be a maximal eigenvalue.
  - m is a __________ vector.

    $(tx)\, m$ must vanish $\quad \forall x \in \mathfrak{g}$
  - Hence $t\mathfrak{g}$ acts as zero on $_\sigma(\tilde{M})$ ... ~~so do $\mathfrak{g}$ & $\hat{\mathfrak{g}}$~~.

·· So $\mathfrak{g}$ and G act on $\sigma(\tilde{M})$. *

·· So $\exists$ G-irrep $V_\nu \hookrightarrow \sigma(\tilde{M})$.

·· By recip. we have $V_\nu^K \xrightarrow[\neq 0]{} \tilde{M}$.

·· So $\tilde{M}$ has a L_0-generalized eigenvector

w/ L_0-gen-eigenvalue $P_K(\nu)$.

·· Since $M_1 \subsetneq V_\lambda^K$, $P_K(\lambda) - P_K(\nu) \in \mathbb{N}_{\geq 1}$.

so $\nu \in S$.

#

§3. Category $\mathcal{O}$

<u>Define</u> (Category $\mathcal{O}$)

Let $\mathcal{O} = \mathcal{O}_\kappa$ be the full subcat of $\underline{KL_\kappa}$ consisting of f.g. $\hat{\mathfrak{g}}_\kappa$-reps.

<u>Prop 7</u> $\left(V \in \mathcal{O} \right) \iff \left(V \text{ is a quotient of a generalized Weyl module} \right)$.

(proof)

- $(\Leftarrow)$:
  - Let V be a gen. Weyl module.
  - By def, ... done.

• ($\Rightarrow$):

•• Let $V \in \mathcal{O}$ i.e.

1) $t\mathfrak{g}[[t]]$ acts loc. nilp. on V.

2) $\mathfrak{g} \curvearrowright V$ integrates to $G \curvearrowright V$.

3) $\exists\ S \subset V$ s.t. $|S| < \infty$ and $\hat{\mathfrak{g}}_k S = V$

•• By 1), $G[[t]]\, S =: M$ is a finite dimensional $G[[t]]$ submodule of $\mathrm{Res}^{\hat{\mathfrak{g}}_k}_{\hat{\mathfrak{g}}_{k,+}}(V)$.

•• By recip. $\underbrace{M}_{\text{Gen Wyl module.}} \xrightarrow{k.} V$ b/c (3)

#

Prop 8 (more equivalent statements) Let $V \in KL_k$. TFAE.

(1) $V(N)$ is finite dimensional and generates V as a $\widehat{\mathfrak{g}_k}$-module for some $N \in \mathbb{N}$.

(2) V is a quotient of a gen. Weyl module $\overset{\text{prop 7}}{(\Longleftrightarrow} V \in \mathcal{O})$

(3) <u>V admits a finite filtration</u> w/ quotients L_ν^k.

recall unique irred quotient of V_ν^k.

Coro 9 $\mathcal{O}_k$ is abelian.

RMK (Theorem, w/o proof in this talk) $(8.1) \Longleftrightarrow V(1)$ is finite dimensional.

(proof)

- $1) \Rightarrow 2)$

  - $\exists\, N \in \mathbb{N}$ s.t. $\overset{①}{\hat{\mathfrak{g}}}_K V(N) = \overset{V}{\cancel{0}}$ and $\dim V(N) < \infty$.

  - We have a $G[t]$-mod map $V(N) \hookrightarrow V$.

  - Reorp $\Rightarrow$ $V(N)^{\kappa} \twoheadrightarrow V$. $\because$ ①

    #

- $2) \Rightarrow 3)$

  - Let V be a quotient of a gen. Weyl module.

  - By Prop 6, it's enough to show that any irred quot Q of V is of the form L_ν^κ for some $\nu \in \Lambda^+$.

∴ By prop 4, it's enough to construct a nontrivial map from $V_\nu^\kappa \longrightarrow Q$ for some $\nu \in \Lambda^+$.

∴ The rest is the same as in the proof of prop 6.

(to ~~gen.~~ construct a nontrivial map)

reciprocity + spectrum bdded above.

$\#$

• (3) ⇒ (1)

•• Let V have a finite filtration w/ quot being L_ν^κ.

(⇒ V is finitely generated.)

•• Since $V \in KL_\kappa$, $t\mathfrak{g}[[t]]$ acts loc. nilp on V.

thus $V = \bigcup_{N \in \mathbb{N}} V(N)$.

∵ f.g. ∴ some $V(N)$ contains a set of generators.

•• It remains to show that $\dim_{\mathbb{C}} V(N) < \infty$.

•• ∃ S.E.S. $0 \to V(1) \to V(N) \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, V(N-1)) \to 0$

$x \mapsto (g \mapsto (tg)x)$

•• By induction, it's enough to prove $\dim V(1) < \infty$.

•• Any SES $0 \to W \to V \to V/W \to 0$ in KL_K

gives ~~an~~ ES $0 \to W(1) \to V(1) \to V/W(1)$.

so it's enough to show ~~that~~ for $W(1)$ & $V/W(1)$

for all $W \leq V$.

•• It remains to show $\dim \underline{L_\lambda^K(1)} < \infty$.

$\cong$

V_λ

$\mathfrak{g}$-mod

by prop 5.

$\#$

§ Duality functor $\mathcal{O}^{op} \xrightarrow[\sim]{\mathbb{D}} \mathcal{O}$

In fact, we'll define $\mathbb{D}$ for a larger cat. "$\mathcal{C}$" and show it restricts to an endofunctor of $\mathcal{O}$.

Define the cat $\mathcal{C}$.

Let $\mathcal{C}$ be the full subcat of KL_K w/ objects V s.t. decomposition in L_0-gen. eigenspaces $\bigoplus_{x \in \mathbb{C}} {}_xV$ satisfies

1) $\dim({}_xV) < \infty$ $\quad \forall x \in \mathbb{C}$

2) $\exists \{x_1, x_2, \dots, x_\ell\} \subseteq \mathbb{C}$

s.t. $\{x \in \mathbb{C} \mid {}_xV \neq 0\} \subseteq \bigcup_{j=1}^{\ell} (x_j - \mathbb{Z}_{\geq 0})$.

RMK $\mathcal{O} \subseteq \mathcal{C}$.

Define $\left(\mathcal{C}^{op} \xrightarrow{D} \mathcal{C} \right)$

- Given $V \in Obj(\mathcal{C})$, define $D(V)$ as follows..
  - as a vect space, $D(V) := \left(\bigoplus_{x \in \mathbb{C}} ({}_xV)^* \right) \subseteq V^*$.
  - as a $\hat{\mathfrak{g}}_K$-module, the action is given as follows

$\tilde{\mathfrak{g}}_K = \mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}\cdot c$

... $\exists$ anti-involution

$$\begin{aligned} \tilde{\mathfrak{g}}_K &\xrightarrow{\ \#\ } \tilde{\mathfrak{g}}_K \\ c &\longmapsto c \\ t^n x &\longmapsto t^{-n} x \qquad \forall x \in \mathfrak{g} \end{aligned}$$

... Let $\tilde{\mathfrak{g}}_K \curvearrowright D(V)$ by the $\#$-twisted dual action:

$$(a.\underset{V^*}{\phi})(\underset{V}{v}) := \phi\big((\# a).v\big).$$

... Since the spectrum is bdd above, the action extends to $\hat{\mathfrak{g}}_K$.

RMK
1) ${}_\chi(D(V)) = ({}_\chi V)^* \quad \forall V \in \mathcal{C}$ (Hence $D(V) \in \mathcal{C}$.)
2) since ${}_\chi V$ is fin dim'l, $D^2(V) = V$. ($\forall V \in \mathcal{C}$)
3) D is exact.

Prop 10 $\mathcal{D}$ restricts to $\mathcal{O}$.

proof

- Let $V \in \mathrm{Obj}(\mathcal{O})$. Goal: $\mathcal{D}(V) \in \mathrm{Obj}(\mathcal{O})$.
- Prop 8.3 $\Leftrightarrow$ V has a finite filt. w/ quot being L^k_ν.

  So it's enough to show that $\mathcal{D}(L^k_\nu) = L^k_\lambda$)

  $\because$ Use 8.3 again ... $\mathcal{D}(V) \in \mathcal{O}$.
- Remains to show $\mathcal{D}(L^k_\nu) = L^k_\lambda$ for some λ.

  (in fact, $\underbrace{V_\lambda^* = V_\nu}_{G\text{-mod}}$).
- Since $\mathcal{D}^2 = \mathrm{id}$, and $\mathcal{D}$ is exact, $\mathcal{D}(L^k_\nu)$ is irreducible.

- We have a G-map: $V_\lambda \hookrightarrow D(L_\nu^k)$

  by the same trick in the proof of Prop 6.

- Recip $\Rightarrow$ $V_\lambda^k \longrightarrow D(L_\nu^k)$

- Prop 4 $\Rightarrow$ $V_\lambda^k \twoheadrightarrow L_\lambda^k \simeq D(L_\nu^k)$.

#.