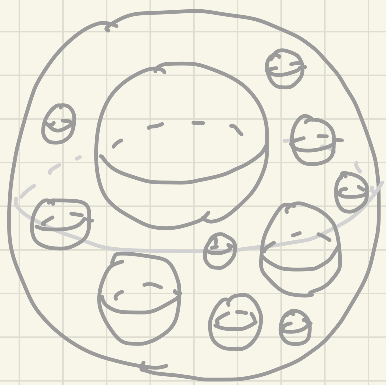
Stony Brook Colloquium

Hee Oh

(Yale)

April 2021

Unipotent flows on hyperbolic manifolds à la Ratner



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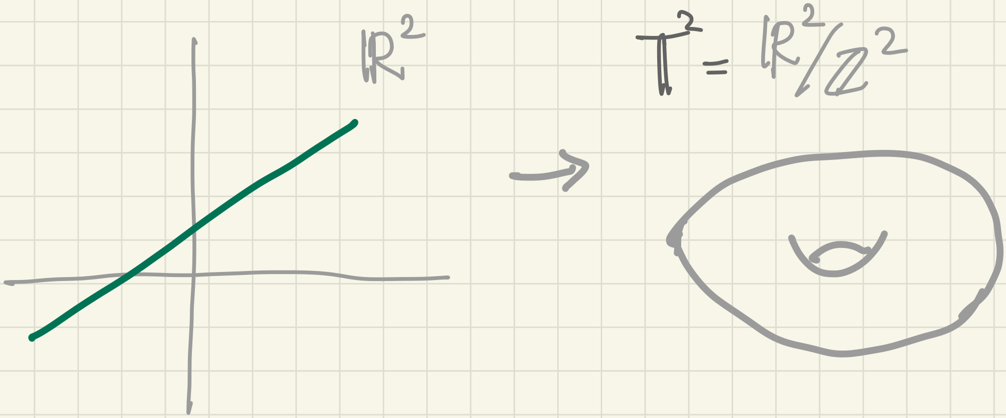
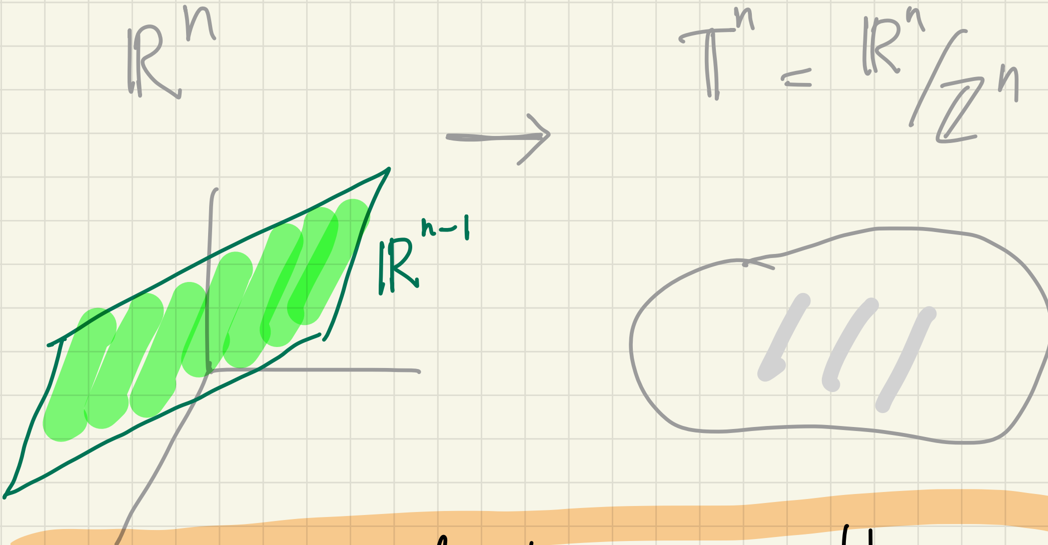
Yale University

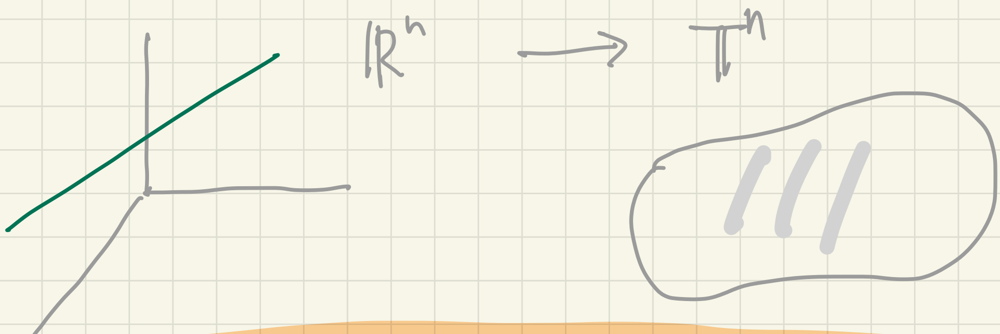
* Lines on the torus



Any line is either closed or dense in $\mathbb{T}^2$



Any maximal plane is either closed or dense in $\mathbb{T}^n$



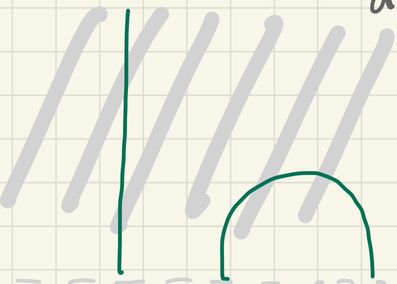
The closure of a line is a k-torus in $\mathbb{T}^n$ for some $1 \leq k \leq n$.

Generalizations
of these phenomena
to hyperbolic manifolds
of finite / infinite
volume.

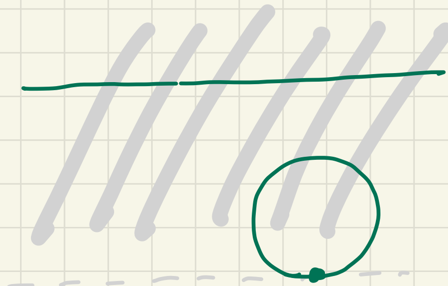
1. Hedlund's thm (1936)

$\mathbb{H}^2 = \{(x,y) \mid y>0\}$ $\quad \partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} = S^1$

$ds = \frac{d_{Euc}}{y}$



geodesics

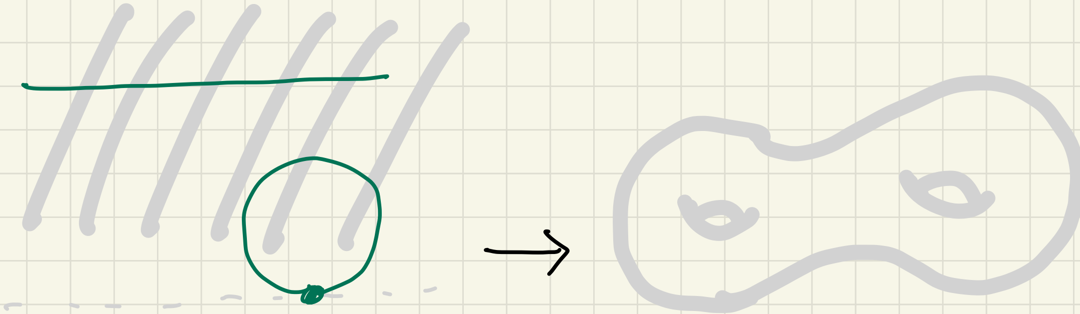


horocycles

$\mathrm{Isom}^+(\mathbb{H}^2) = PSL_2\mathbb{R}$

Any complete hyp. surface $= \Gamma \backslash \mathbb{H}^2$,
(closed)

$\Gamma < PSL_2\mathbb{R}$ co-cpt discrete subgp



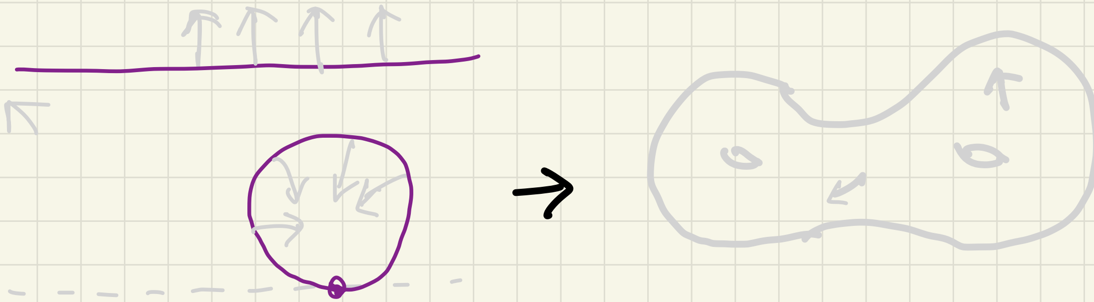
$\mathbb{H}^2$ $\qquad\qquad$ $S = \Gamma \backslash \mathbb{H}^2$

Hedlund (1936) $\quad S$: closed hyp. surface

Any horocycle is dense in S

__Rmk__ Not true for geodesics

$T^1(\mathbb{H}^2) = PSL_2\mathbb{R}$ $\qquad$ $T^1(S) = \Gamma\backslash PSL_2\mathbb{R}$



$$U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

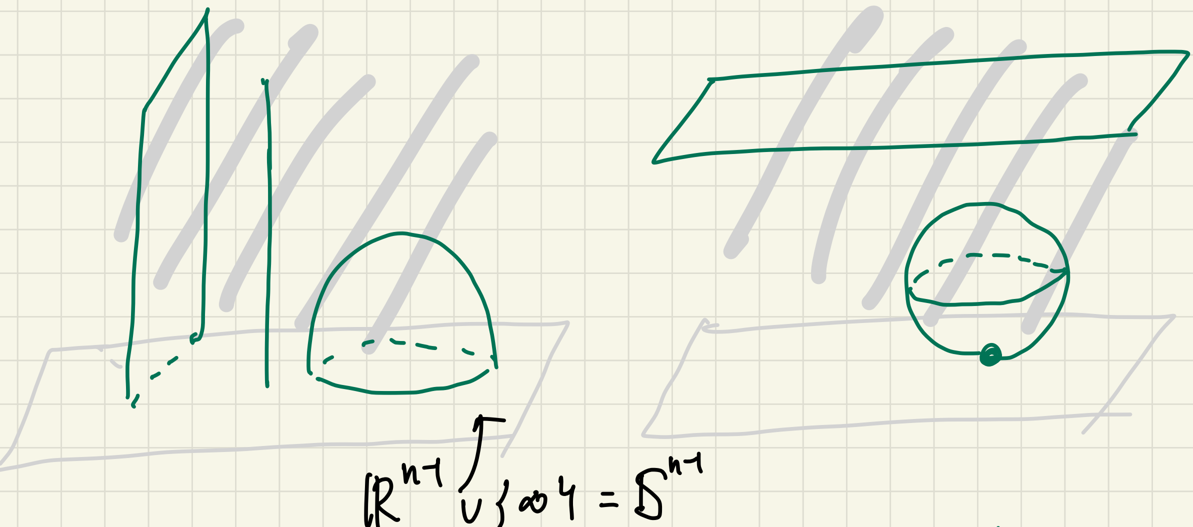
horocycles $\leftarrow$ U-orbits

Hedlund's thm

$\forall x \in \Gamma\backslash PSL_2\mathbb{R}$, $\quad \overline{xU} = \Gamma\backslash PSL_2\mathbb{R}$

2. Veech's thm (1975)

$\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, y) \mid y > 0\}$, $ds = \dfrac{d_{Euc}}{y}$



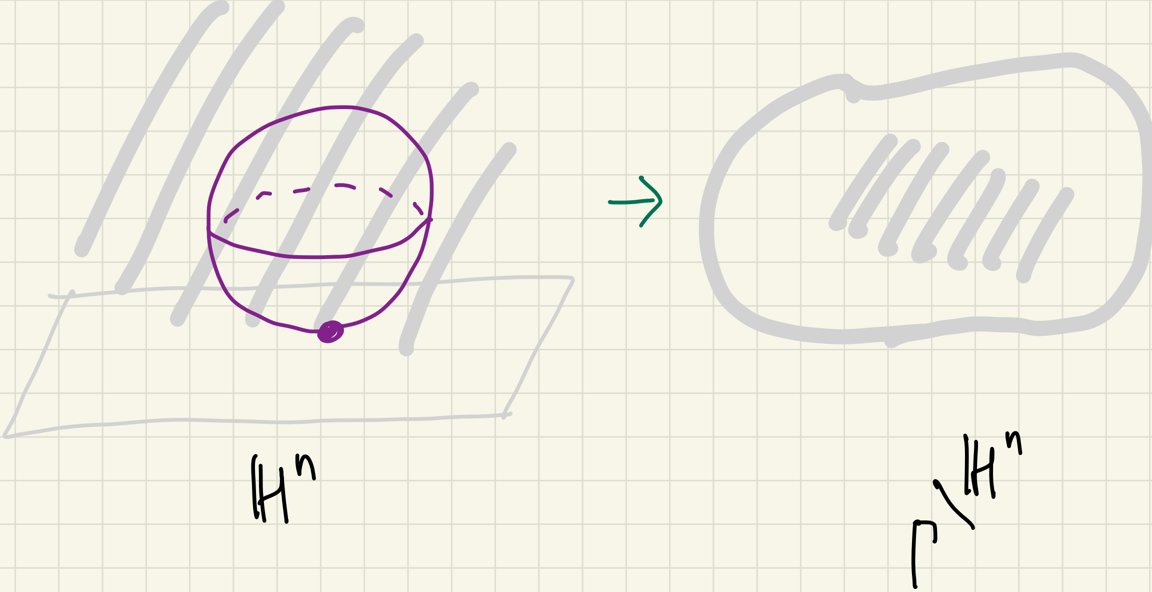
geodesic k-planes

maximal horospheres

$\mathrm{Isom}^+(\mathbb{H}^n) = SO^\circ(n,1)$

Complete hyp. n-mflds $= \Gamma \backslash \mathbb{H}^n$
(closed)

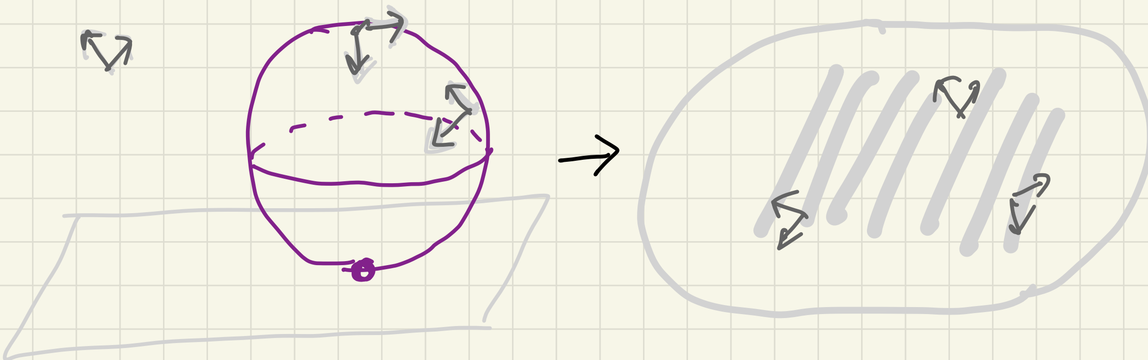
$\Gamma < SO^\circ(n,1)$
(cocpt) discrete subgp



Veech (1975) M : closed hyp n-mfld

Any max. horosphere is dense in M

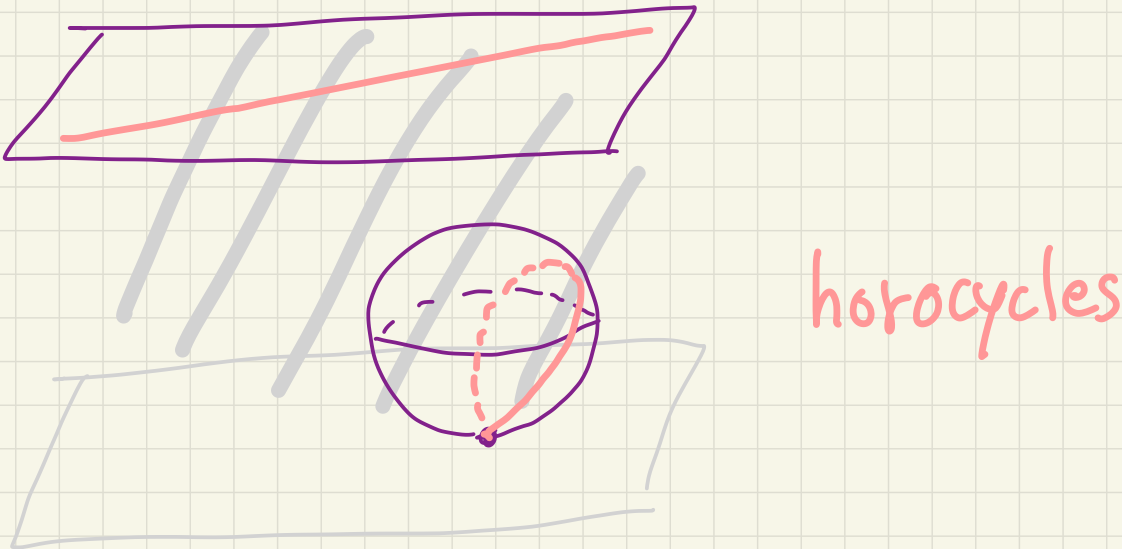
$$F(\mathbb{H}^n) = SO^\circ(n,1) \qquad F(M) = \Gamma\backslash SO^\circ(n,1)$$



$$U = \left\{ \begin{pmatrix} 1 & x_1, \cdots, x_{n-1} & -\frac{1}{2}\sum x_i^2 \\ 0 & & -x_1 \\ \vdots & I_{n-1} & \vdots \\ 0 & & -x_{n-1} \\ 0 & 0 \;\cdots\; 0 & 1 \end{pmatrix} \right\} \cong \mathbb{R}^{n-1}$$

max. horospheres $\leftarrow$ U-orbits.

$$\boxed{\forall\, x \in \Gamma\backslash SO^\circ(n,1), \quad \overline{xU} = \Gamma\backslash SO^\circ(n,1)}$$



What are the closures of horocycles in $\Gamma \backslash \mathbb{H}^n$?

3. Ratner's thm (1991)

G : conn semi-simple linear Lie gp

(e.g $SL_n\mathbb{R}$, $SO^\circ(n,1)$, ...)

$\Gamma < G$ lattice (= discrete subgp of finite co-vol)

(e.g $SL_n\mathbb{Z} < SL_n\mathbb{R}$)

$\Gamma \backslash G \curvearrowleft H$: conn subgp gen by unipotent elements

Thm (Ratner) conj. by Raghunathan

$\forall x \in \Gamma \backslash G$, $\overline{xH} = xL$

where $H < L < G$

L: conn. closed subgp

Special case

$$\boxed{\text{Any } SO^\circ(2,1)\text{-orbit in } SL_3\mathbb{R}/SL_3\mathbb{Z} \text{ is closed or dense.}}$$

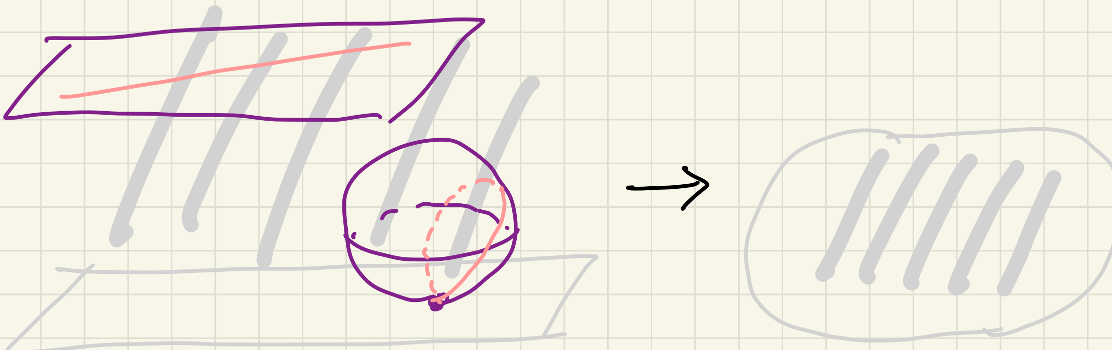
$\Rightarrow$ Oppenheim Conjecture (1929)

proved by Margulis (1987)

Q : irrational indef. quad form in $n \geq 3$ variables.

$\Rightarrow 0 \in \overline{Q(\mathbb{Z}^n \setminus \{0\})}$

Special case



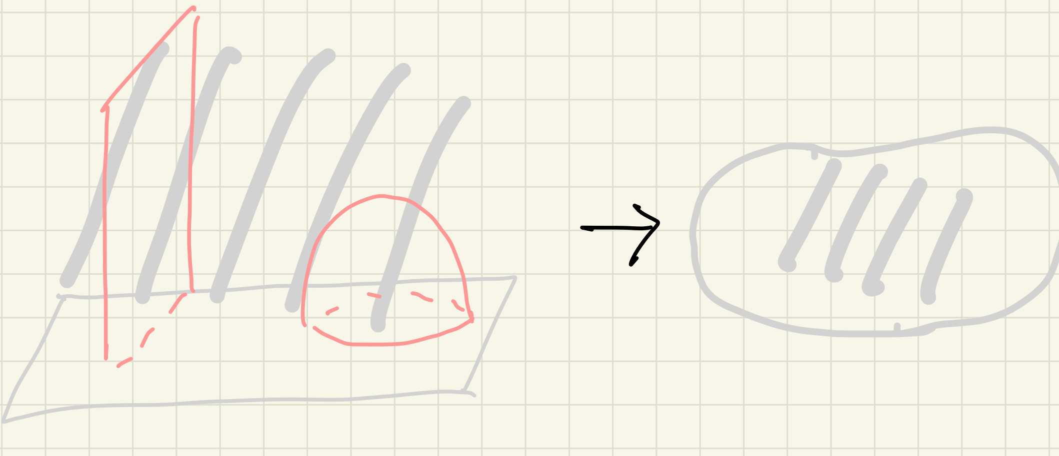
$M = \Gamma \backslash \mathbb{H}^n$ hyp mfld with $\mathrm{vol}(M) < \infty$

- $\overline{\text{horocycle}}$ = properly immersed submfld

$\Gamma \backslash SO^\circ(n,1) \circlearrowleft H$ = one-dim'l subgp of

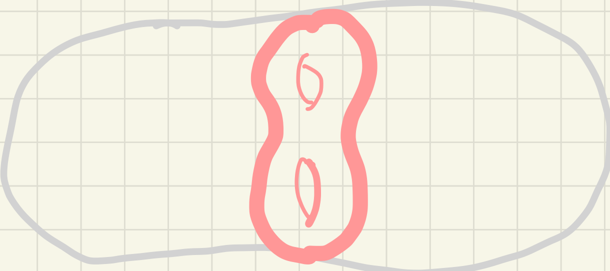
$$U = \begin{pmatrix} 1 & x_1 & \cdots & x_{n-1} & -\frac{1}{2}\sum x_i^2 \\ 0 & & & & -x_1 \\ \vdots & & I_{n-1} & & \vdots \\ \vdots & & & & -x_{n-1} \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

- $\overline{\text{geod. plane}}$ = properly immersed submfld

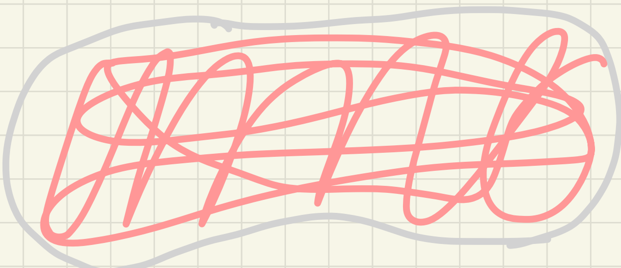


geod k-plane $\leftarrow$ orbits of $SO^\circ(k,1)$ in $\Gamma \backslash SO^\circ(n,1)$

For $n=3$, any geod plane
is closed or dense



closed

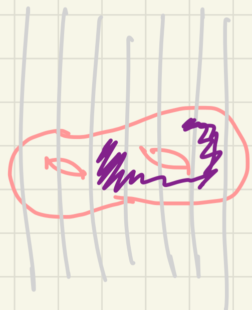


dense

For $n \geqslant 4$, there may be
intermediate cases.

4. Does Ratner's thm still hold in ∞ - vol setting ?

<u>No</u> for certain hyp 3-mflds $\approx \Sigma \times \mathbb{R}$,



some geod planes have wild closures

(McMullen-Mohammadi-O.)

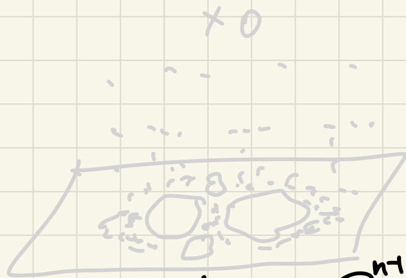
<u>Yes</u> for convex cocpt hyp mflds with Fuchsian ends

$\Gamma < SO^{\circ}(n,1) = G$ Zariski dense

$M = \Gamma \backslash \mathbb{H}^n$

Def

- The limit set Λ

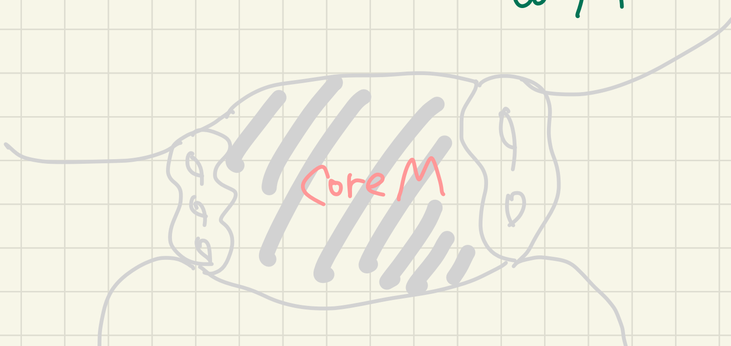


$\mathbb{R}^{n-1} \cup \{\infty\} = S^{n-1}$

- $\text{Core } M = \Gamma \backslash \text{hull}(\Lambda)$

$\subset \Gamma \backslash \mathbb{H}^n = M$

Smallest convex submfld homotopic to M

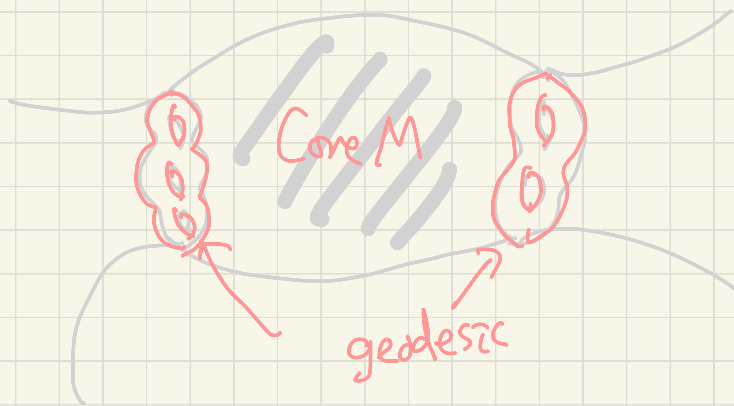


$M = \Gamma \backslash \mathbb{H}^n$

Assume M is convex compact

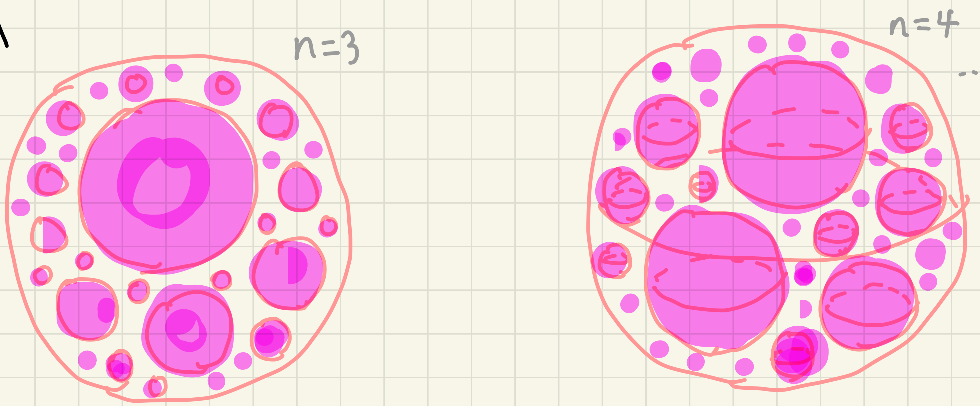
($=$ Core M is compact)

Def M has Fuchsian ends if $\partial\,\mathrm{Core}\,M =$ totally geodesic

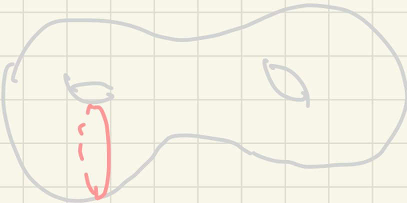


$\mathbb{B}^{n-1} - \Lambda$

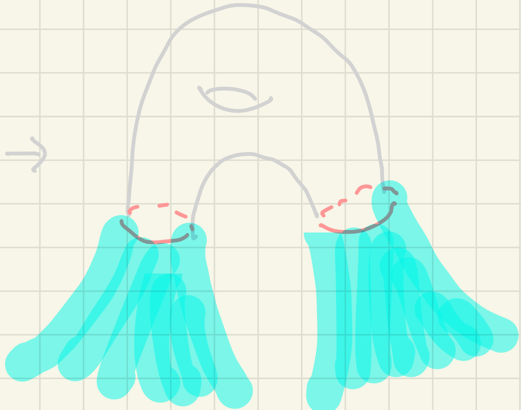
$=$



For $n=2$,

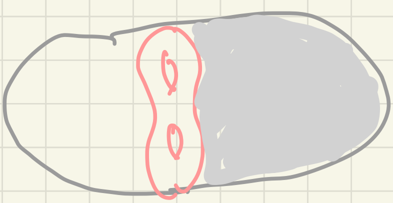


simple closed geod



For $n \geq 3$

closed hyp mflds $\supset$ closed hyp mflds with prop. embedded co-dim 1 geod plane $\rightsquigarrow$ convex cocpt hyp mflds with Fuchsian ends



Thm M : convex cocpt
hyp $n \geq 3$ - mfld
Fuchsian ends

$\overline{\text{horocycle}}$

are properly immersed sub-mflds.

$\overline{\text{geod. plane}}$

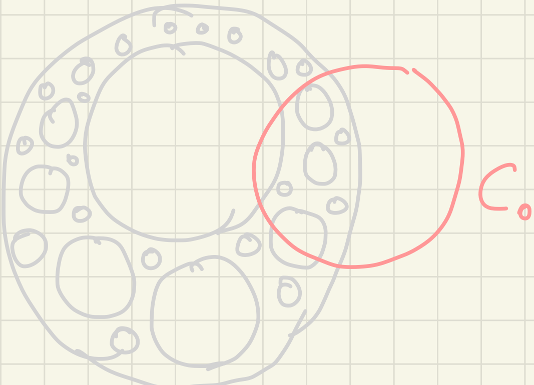
$$\left(\begin{array}{ll} n=3. & \text{McMullen-Mohammadi-O.} \\ n \geq 4 & \text{Lee-O.} \end{array} \right)$$

Orbits of circles

$$S^{n-1} - \Lambda = \bigcup B_i \qquad \overline{B_i} \cap \overline{B_j} = \phi$$

round balls

$n=3$



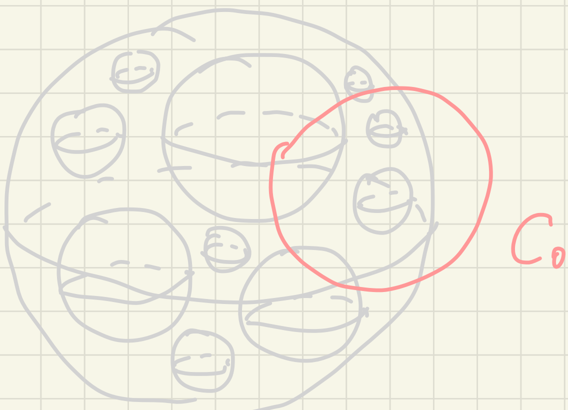
$$\overline{\Gamma(C_0)} = \{ C \subset \Gamma S_0 \mid C \cap \Lambda \neq \phi \}$$

where

S_0 : k-sphere

$\Gamma(S_0)$ closed

$n=4$



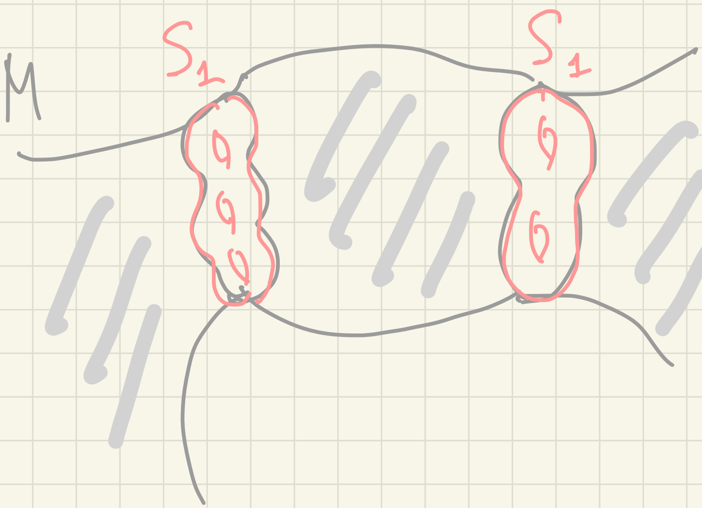
$\vdots$

$n \geq 3$

c'bly many closed hyp n-mfld $\rightsquigarrow$ hyp n-mflds with Fuchsian ends

$n=3$ $\to$ huge def. space of hyp 3-mflds with quasi-Fuchsian ends

$n \geq 4$ $\to$ No local deformations.



$\mathbb{R}^{6g_1-6} \times \mathbb{R}^{6g_2-6}$

$=$ Teich$(S_1) \times$ Teich(S_2)

$\updownarrow$

deformations of M

Thm (McMullen-Mohammadi-O.)

M: convex cocpt hyp 3-mfld with quasi-Fuchsian ends

$M^* =$ interior of Core M

geod. plane $\cap M^*$ is closed or dense in M^*

<u>Rmk</u> Cannot Replace M^* by M

(ex. by Zhang)

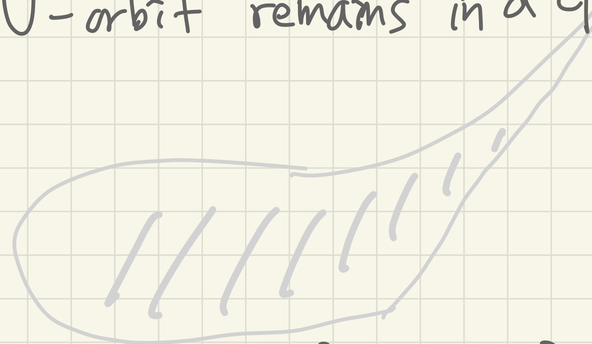
1st difficulty in carrying out unip. dynamics in inf-volume setting

$$U=\{u_t \mid t\in\mathbb{R}\}$$

$\Gamma\backslash G$ cpt

Any U-orbit remains in a cpt set

$\mathrm{vol}(\Gamma\backslash G)<\infty$



Any U-orbit spends 99% time in a cpt subset

$\mathrm{vol}(\Gamma\backslash G)=\infty$

Almost all U-orbit spends 0% time in a cpt subset.

If M has Fuchsian ends,

$\exists$ cpt subset $\Omega \subset \frac{G}{\Gamma}$ s.t

$\{t \in \mathbb{R} \mid x u_t \in \Omega\}$

$\sim$ thick Cantor set.

Moreover

$\{t \in \mathbb{R} \mid x u_t \in \Omega -$ "nbd. of singular set" $\}$

$\sim$ thick Cantor set

Thank

you !