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Orbital counting via Mixing and Unipotent flows

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This note is an expanded version of my lectures given at the Clay summer school in 2007.

1. Introduction: motivation

Let $\mathbf{X}$ be a projective algebraic variety defined over $\mathbb{Q}$, that is, $\mathbf{X}$ is the set of (equivalence classes of) zeros of homogeneous polynomials with coefficients in $\mathbb{Q}$. The set $\mathbf{X}(\mathbb{Q})$ of rational points in $\mathbf{X}$ consists of rational zeros of the polynomials.

The following is a classical question in number theory:

"understand the set $\mathbf{X}(\mathbb{Q})$ of rational points".

More detailed questions can be formulated as follows:

(1) Is $\mathbf{X}(\mathbb{Q})$ non-empty?
(2) If non-empty, is $\mathbf{X}(\mathbb{Q})$ infinite?
(3) If infinite, is $\mathbf{X}(\mathbb{Q})$ Zariski dense in $\mathbf{X}$?
(4) If Zariski dense, setting

$$\mathbf{X}_T := \{x \in \mathbf{X}(\mathbb{Q}) : \text{“ size”}(x) < T\},$$

what is the asymptotic growth rate of $\#\mathbf{X}_T$ as $T \to \infty$?

(5) Interpret the asymptotic growth rate of $\#\mathbf{X}_T$ in terms of geometric invariants of $\mathbf{X}$
(6) Describe the asymptotic distribution of $\mathbf{X}_T$ as $T \to \infty$?

A basic principle in studying these questions is that

(1.1) "the geometry of $\mathbf{X}$ governs the arithmetic of $\mathbf{X}$".

A good example demonstrating this philosophy is the Mordell conjecture proved by Faltings [**30**]:

THEOREM 1.2. *If* $\mathbf{X}$ *is a curve of genus at least* 2*, then* $\mathbf{X}(\mathbb{Q})$ *is finite.*

Note in the above theorem that a purely geometric property of $\mathbf{X}$ imposes a very strong restriction on the set $\mathbf{X}(\mathbb{Q})$.

Smooth projective varieties are classified roughly into three categories according to the ampleness of its canonical class $\mathrm{K}_{\mathbf{X}}$. For varieties of general type ($\mathrm{K}_{\mathbf{X}}$ ample), the following conjecture by Bombieri, Lang and Vojta is a higher dimensional analogue of the Mordell conjecture [**65**]:

CONJECTURE 1.3. *If* $\mathbf{X}$ *is a smooth projective variety of general type, then* $\mathbf{X}(K)$ *is not Zariski dense for any number field* K.

At the other extreme are Fano varieties ($-\mathrm{K}_{\mathbf{X}}$ ample). It is expected that a Fano variety should have a Zariski dense subset of rational points, at least after passing to a finite field extension of $\mathbb{Q}$. Moreover Manin formulated conjectures in 1987 on the questions (4) and (5) for some special "size functions", which were soon generalized by Batyrev and Manin for more general size functions [**1**], and Peyre made a conjecture on the question (6) [**50**].

In these lecture notes, we will focus on the question (4). First, we discuss the notion of the *size* of a rational point in $\mathbf{X}(\mathbb{Q})$. One way is simply to look at the Euclidean norm of the point. But this usually gives us infinitely many points of size less than T, and does not encode enough arithmetic information of the point.

Height function— A suitable notion of the size of a rational point is given by a height function. For a general variety $\mathbf{X}$ over $\mathbb{Q}$ and to every line bundle L of $\mathbf{X}$ over $\mathbb{Q}$, one can associate a height function H_{L} on $X(\mathbb{Q})$, unique up to multiplication by a bounded function, via Weil's height machine (cf. [**38**]). On the projective space $\mathbb{P}^n$, the height $\mathrm{H} = \mathrm{H}_{\mathcal{O}_{\mathbb{P}^n(1)}}$ associated to the line bundle $\mathcal{O}_{\mathbb{P}^n(1)}$ of a hyperplane is defined by:

$$\mathrm{H}(x) := \sqrt{x_0^2 + \cdots + x_n^2}$$

where $(x_0, \cdots, x_n)$ is a primitive integral vector representing $x \in \mathbb{P}^n(\mathbb{Q})$, which is unique up to sign.

It is clear that there are only finitely many rational points in $\mathbb{P}^n(\mathbb{Q})$ of height less than T, as there are only finitely many (primitive) integral vectors of Euclidean norm at most T. Schanuel [**57**] showed that as $T \to \infty$,

$$\#\{x \in \mathbb{P}^n(\mathbb{Q}) : \mathrm{H}_{\mathcal{O}_{\mathbb{P}^n(1)}}(x) < T\} \sim c \cdot T^{n-1}$$

for an explicit constant $c > 0$, and this is the simplest case of Manin's conjecture.

For a general variety $\mathbf{X}$ over $\mathbb{Q}$, a very ample line bundle L of $\mathbf{X}$ over $\mathbb{Q}$ defines a $\mathbb{Q}$-embedding $\psi_{\mathrm{L}} : \mathbf{X} \to \mathbb{P}^n$. Then a height function H_{L} is the pull back of the height function $\mathrm{H}_{\mathcal{O}_{\mathbb{P}^n(1)}}$ to $\mathbf{X}(\mathbb{Q})$ via ψ_{L}. For an ample line bundle L, we set

$$\mathrm{H}_{\mathrm{L}} = \mathrm{H}_{\mathrm{L}^k}^{1/k}$$

for $k \in \mathbb{N}$ such that L^k is ample.

Note that for any ample line bundle L,

$$\#\{x \in \mathbf{X}(\mathbb{Q}) : \mathrm{H}_\mathrm{L}(x) < T\} < \infty.$$

For a subset U of $\mathbf{X}$ and $T > 0$, we define

$$N_U(L, T) := \#\{x \in \mathbf{X}(\mathbb{Q}) \cap U : \mathrm{H}_\mathrm{L}(x) < T\}.$$

Manin's conjecture— Manin's conjecture (or more generally the Batyrev-Manin conjecture) says the following (cf. [**1**]):

CONJECTURE 1.4. *Let* $\mathbf{X}$ *be a smooth Fano variety defined over* $\mathbb{Q}$. *For any ample line bundle* L *of* $\mathbf{X}$ *over* $\mathbb{Q}$, *there exists a Zariski open subset* U *of* $\mathbf{X}$ *such that (possibly after passing to a finite field extension), as* $T \to \infty$,

$$N_U(L, T) \sim c \cdot T^{a_\mathrm{L}} \cdot (\log T)^{b_\mathrm{L} - 1}$$

where $a_\mathrm{L} \in \mathbb{Q}_{>0}$ *and* $b_\mathrm{L} \in \mathbb{Z}_{\geq 1}$ *depend only on the geometric invariants of* L *and* $c = c(\mathrm{H}_\mathrm{L}) > 0$.

More precisely, the constants a_L and b_L are given by:

$a_\mathrm{L} := \inf\{a : a[\mathrm{L}] + [\mathrm{K}_\mathbf{X}] \in \Lambda_{\mathrm{eff}}(\mathbf{X})\}$;

$b_\mathrm{L} :=$ the codimension of the face of $\Lambda_{\mathrm{eff}}(\mathbf{X})$ containing $a_\mathrm{L}[\mathrm{L}] + [\mathrm{K}_\mathbf{X}]$ in its interior

where $\mathrm{Pic}(\mathbf{X})$ denotes the Picard group of $\mathbf{X}$ and $\Lambda_{\mathrm{eff}}(\mathbf{X}) \subset \mathrm{Pic}(\mathbf{X}) \otimes \mathbb{R}$ is the cone of effective divisors.

REMARK 1.5. *The restriction to a Zariski open subset is necessary in Conjecture 1.4: for the cubic surface* $\mathbf{X}$: $\sum_{i=0}^3 x_i^3 = 0$, *the above conjecture predicts the* $T(\log T)^3$ *order of rational points of height* $\sqrt{\sum_{i=0}^3 x_i^2} < T$, *but the curve* $\mathbf{Y}$ *given by the equations* $x_0 = -x_1$ *and* $x_2 = -x_3$ *contains the* T^2 *order of rational points of height bounded by* T.

Note that the asymptotic growth of the number of rational points of bounded height, which is arithmetic information on $\mathbf{X}$, is controlled only by the geometric invariants of $\mathbf{X}$; so this conjecture as well embodies the basic principle (1.1) for Fano varieties.

Conjecture 1.4 has been proved for smooth complete intersections of small degree [**6**], flag varieties [**31**], smooth toric varieties [**2**], smooth equivariant compactifications of horospherical varieties [**60**], smooth equivariant compactifications of vector groups [**11**], smooth bi-equivariant compactifications of unipotent groups [**59**] and wonderful compactifications of semisimple algebraic groups ([**58**] and [**33**]). We refer to survey articles ([**62**], [**63**], [**10**]) for more backgrounds.

In the first part of these notes, we will discuss a recent work of Gorodnik and the author [**35**] which solves new cases of Conjecture 1.4 for certain compactifications of homogeneous varieties. In contrast to most of the previous works which were based on the harmonic analysis on the corresponding adelic spaces in order to establish analytic properties of the associated height zeta function, our approach is to use the dynamics of flows on the homogeneous spaces of adele groups.

Approach—We will be interested in the projective variety $\mathbf{X}$ which is the compactification of an affine homogeneous variety $\mathbf{U}$, and try to understand the asymptotic of the number of rational points of $\mathbf{U}$ of height less than T (note that $\mathbf{U}$ is a Zariski open subset of $\mathbf{X}$ and hence it suffices to count rational points lying in $\mathbf{U}$.). More precisely, let $\mathbf{U}$ be an orbit $u_0\mathbf{G}$

where $\mathbf{G} \subset \mathrm{PGL}_{n+1}$ is an algebraic group defined over $\mathbb{Q}$ and $u_0 \in \mathbb{P}^n(\mathbb{Q})$. And let $\mathbf{X} \subset \mathbb{P}^n$ be the Zariski closure of $\mathbf{U}$, and consider the height function H on $\mathbf{X}(\mathbb{Q})$ obtained by the pull pack of $\mathrm{H}_{\mathcal{O}_{\mathbb{P}^n}(1)}$.

We attempt to forget about the ambient geometric space $\mathbf{X}$ for the time being and to focus on the rational points $\mathbf{U}(\mathbb{Q})$ of the affine homogeneous variety $\mathbf{U}$.

We would like to prove that

$$N_T := \#\{x \in \mathbf{U}(\mathbb{Q}) : \mathrm{H}(x) < T\} \sim c \cdot T^a \cdot (\log T)^{b-1} \tag{1.6}$$

for some $a, c > 0$ and $b \geq 1$. How does one prove such a result? Or where should the growth rate $T^a (\log T)^{b-1}$ come from?

Consider for a moment how to count integral vectors of Euclidean norm less than T in the plane. How does one know that the asymptotic of the number N_T of such integral vectors is of the form πT^2, as $T \to \infty$? It is because that one can show that N_T is asymptotic to the area of disc of radius T and compute that the area is πT^2, using calculus.

It turns out that one can follow the same basic strategy for counting rational points. We will first understand the set $\mathbf{U}(\mathbb{Q})$ of rational points as a discretely imbedded subset in certain ambient locally compact space and show that N_T is asymptotic to the volume of a suitably defined *height ball* in this ambient space.

What is this ambient space where the set $\mathbf{U}(\mathbb{Q})$ can be put as a discrete subset? In the real algebraic variety $\mathbf{U}(\mathbb{R})$, why is $\mathbf{U}(\mathbb{Q})$ not discrete? It is because the denominators of points can tend to infinity along prime numbers. The resolution of this issue can be found precisely using the language of *adeles*. In section 2, we define the adeles and discuss their basic properties.

Once we have defined the adelic space $\mathbf{U}(\mathbb{A})$ which contains $\mathbf{U}(\mathbb{Q})$ as a discrete subset, we will be extending the height function H of $\mathbf{X}(\mathbb{Q})$ to a continuous proper function on $\mathbf{U}(\mathbb{A})$, which we again denote by H, so that $B_T := \{x \in \mathbf{U}(\mathbb{A}) : \mathrm{H}(x) \leq T\}$ is a compact subset of $\mathbf{U}(\mathbb{A})$. Then

$$\{x \in \mathbf{U}(\mathbb{Q}) : \mathrm{H}(x) \leq T\} = \mathbf{U}(\mathbb{Q}) \cap B_T.$$

Our techniques based on the dynamical approach work for the *orbital* counting function. That is, we will be looking only at a single $\mathbf{G}(\mathbb{Q})$-orbit in $\mathbf{U}(\mathbb{Q})$ at a time; fixing a $\mathbf{G}(\mathbb{Q})$-orbit $\mathcal{O} := u_0\mathbf{G}(\mathbb{Q})$ for $u_0 \in \mathbf{U}(\mathbb{Q})$, what can we say about

$$N_T(\mathcal{O}) := \# u_0\mathbf{G}(\mathbb{Q}) \cap B_T?$$

Being able to count points in each orbit $u_0\mathbf{G}(\mathbb{Q})$ is good news and bad news at the same time; it gives finer information on the rational points $\mathbf{U}(\mathbb{Q})$, but does not quite say about the behavior of the total $\mathbf{U}(\mathbb{Q})$, since there are oftentimes infinitely many $\mathbf{G}(\mathbb{Q})$-orbits in $\mathbf{U}(\mathbb{Q})$.

Denote by $\mathbf{L}$ the stabilizer subgroup of u_0 in $\mathbf{G}$. To summarize, we will have a discrete $\mathbf{G}(\mathbb{Q})$-orbit $u_0\mathbf{G}(\mathbb{Q}) = \mathbf{L}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})$ in the homogeneous space $u_0\mathbf{G}(\mathbb{A}) = \mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A}) \subset \mathbf{U}(\mathbb{A})$ and we would like to count points of the orbit $u_0\mathbf{G}(\mathbb{Q})$ in a growing sequence of compact subsets B_T of $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$.

In section 3, we explain a general strategy due to Duke-Rudnick-Sarnak [**21**] on the orbital counting problem for $[x_0]\Gamma \cap B_T$ in a homogeneous space $L\backslash G$; here Γ is a lattice in a second countable locally compact group G, $[x_0]\Gamma$ is discrete in $L\backslash G$ for $[x_0] = L$, and $\{B_T \subset L\backslash G\}$ is a family of compact subsets. This method reduces the counting problem for $N_T(\mathcal{O})$ into understanding

(i) the asymptotic behavior of adelic periods $\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g_i$, as $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$;
(ii) certain regularity problem for the volume of adelic height balls $B_T \subset \mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$.

The techniques needed to establish (ii) are completely disjoint from those for (i), and this is where an input of algebraic/arithmetic geometry is needed. In section 4, we present the main ergodic result on the translates of semisimple periods in the adelic homogeneous space based on the study of unipotent flows and establish (i) when both $\mathbf{G}$ and $\mathbf{L}$ are connected semisimple and $\mathbf{L}$ is a maximal $\mathbb{Q}$-subgroup of G. In section 5, we explain how to deduce some cases of Manin's conjecture using this approach, which is the main result of [**35**]. We also explain the implications of this result on the rational points of *affine* varieties.

In section 6, we deduce mixing theorems for adelic groups as a special case of the results in section 4. We also explain in this section how the equidistribution of Hecke points is related to the adelic mixing theorem. In section 7, we interpret a quantitative adelic mixing theorem as a bound toward the Ramanujan conjecture on the automorphic spectrum. In section 8, we explain how the mixing theorems can be used to prove the equidistribution of symmetric periods, based on the special geometric property of an affine symmetric space, called the wavefront property. This approach gives an effective counting for the S-integral points on affine symmetric varieties. In the last section 9, we discuss a problem of Linnik on the representations of integers by an invariant polynomial, and extend the main result of [**29**] using the ergodic result presented in section 4.

We have not sought at all to state the most general statements. On the contrary, we will be speaking only on the simplest cases in many occasions, for instance, we stick to the field of *rational numbers* $\mathbb{Q}$, rather than a number field, and to the homogeneous spaces $\mathbf{L}\backslash\mathbf{G}$ with $\mathbf{L}$ being a *maximal* connected $\mathbb{Q}$-subgroup of $\mathbf{G}$.

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2. Adeles: definition and basic properties

In this section, we define the notion of adeles, and state some of their basic properties as we would need (cf. [**66**], [**51**]).

2.1. Restricted topological product. —— Let X_p be a second countable locally compact topological space for each p in a given countable index set I. The goal is to construct a reasonable locally compact space which contains all X_p, $p \in I$. The first attempt will be simply taking the direct product $\prod_p X_p$. However the direct product topology does not yield a locally compact space unless either I is finite or almost all X_p are compact.

Hence we will do something else which will make the product a locally compact space and this can be achieved by taking the restricted topological product. Suppose that an open compact subset $K_p \subset X_p$ is given for each p in a co-finite subset I_0 of I.

Set

$$X_I := \prod_p (X_p : K_p) = \{(x_p)_{p\in I} \in \prod_p X_p : x_p \in K_p \text{ for almost all } p\}.$$

We endow on X_I the topology generated by subsets of the form $\prod_{p\in I} V_p$ where V_p is open in X_p and $V_p = K_p$ for almost all p. The space X with this topology is called the restricted topological product of X_p's with respect to the distinguished open subsets K_p's. In the case when I is a finite set, X_I is simply the direct product of X_p, $p \in I$.

We need the following basic properties of X_I:

Facts 2.1. (1) X_I is a second countable locally compact space.

(2) Any compact subset of X_I is contained in
$$X^S := X_S \times \prod_{p\notin S} K_p$$
for some finite $S \subset I$.

(3) If μ_p's are Borel measures on X_p's such that $\mu_p(K_p) = 1$ whenever $p \in I_0$, then the restricted product $\mu := \otimes^*_{p\in I}\mu_p$ on X_I is defined as follows: first for each finite subset $S \subset I$, define the measure μ^S on X^S to be simply the direct product $\prod_{p\in S}\mu_p \times \prod_{p\notin S}\mu_p|_{K_p}$.

Now for any $f \in C_c(X_I)$ whose support is contained in X^S, set
$$\mu(f) := \mu^S(f).$$
It is easy to check that μ is well-defined since μ^S's are compatible with each other. Since X_I can be written as the union $\cup_S X^S$ over finite subsets $S \subset I$, μ defines a Borel measure on X_I by the Riesz representation theorem.

(4) If each X_p is a group (resp. ring), X_I is a group (resp. ring) using the componentwise operations. If μ_p is a (resp. left invariant) Haar measure of X_p for each $p \in I$, then μ is a (resp. left-invariant) Haar measure of X_I.

2.2. $\mathbb{Q}$ is a lattice in the adele group $\mathbb{A}$. —— Denote by $R = \{\infty, 2, 3, \cdots\}$ the set of all primes including the infinite prime ∞. For $p = \infty$, $|\cdot|_\infty$ denotes the usual absolute value on $\mathbb{Q}$ and for p finite, $|\cdot|_p$ denotes the normalized p-adic absolute value on $\mathbb{Q}$, i.e.,
$$\left|p^k \frac{a}{b}\right|_p = p^{-k}$$
if p does not divide ab. We obtain the locally compact fields $\mathbb{R} = \mathbb{Q}_\infty$ and $\mathbb{Q}_p$'s by taking the completions of $\mathbb{Q}$ with respect to $|\cdot|_p$'s. We set
$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \le 1\}$$
for each $p \in R_f := R - \{\infty\}$.

Definition 2.2. The adele ring $\mathbb{A}$ over $\mathbb{Q}$ is defined to be the restricted topological product of $\mathbb{Q}_\infty := \mathbb{R}$ and $(\mathbb{Q}_p, \mathbb{Z}_p)$'s for $p \in R_f$, that is,
$$\mathbb{A} := \prod_{p\in R}(\mathbb{Q}_p : \mathbb{Z}_p).$$

Since every element of $\mathbb{Q}$ belongs to $\mathbb{Z}_p$ for almost all p, $\mathbb{Q}$ can be considered as a subset of $\mathbb{A}$ under the diagonal embedding. That $\mathbb{Q}$ is a discrete subset of $\mathbb{A}$ follows from the following observation, which shows that 0 is an isolated point in $\mathbb{A}$:
$$\{x \in \mathbb{Q} : |x|_\infty < 0.5,\ x \in \mathbb{Z}_p \text{ for all } p \in R_f\} = \{0\}.$$

Moreover $\mathbb{Q}$ is a lattice in $\mathbb{A}$, that is, the quotient of $\mathbb{A}$ by $\mathbb{Q}$ has finite volume (with respect to a Haar measure of $\mathbb{A}$). To show this, we consider the ring $\mathbb{A}_f$ of finite adeles, i.e., the subring of $\mathbb{A}$ whose ∞-component is trivial. Note that $\mathbb{A}_f = \prod_{p\in R_f}(\mathbb{Q}_p : \mathbb{Z}_p)$, and that every element of $\mathbb{A}$ can be written as (x_∞, x_f) with $x_\infty \in \mathbb{R}$ and $x_f \in \mathbb{A}_f$. We set $\mathbb{Z}_f = \prod_{p\in R_f}\mathbb{Z}_p$.

Lemma 2.3. (1) *$\mathbb{Q}$ is dense in $\mathbb{A}_f$.*

(2) $\mathbb{A} = \mathbb{Q} + ((0, 1] \times \mathbb{Z}_f)$.

(3) *$\mathbb{Q}$ is a lattice in $\mathbb{A}$.*

Proof. For (1), it suffices to show that $\mathbb{Z}$ is dense in $\mathbb{Z}_f$. Any open subset in $\mathbb{Z}_f$ contains a subset of the form $\prod_{p\in S}(a_p + p^{m_p}\mathbb{Z}_p) \times \prod_{p\notin S} \mathbb{Z}_p$ for some finite subset S. Hence (1) follows from the Chinese remainder theorem, which says that there exists $x \in \mathbb{Z}$ such that $x = a_p \bmod p^{m_p}$ for all $p \in S$.

(1) implies that $\mathbb{A}_f = \mathbb{Q} + \mathbb{Z}_f$, as $\mathbb{Z}_f$ is an open subgroup of $\mathbb{A}_f$. Now for $(x_\infty, x_f) \in \mathbb{A}$, we have $x_f = y_f + z \in \mathbb{Z}_f + \mathbb{Q}$. Hence $(x_\infty, x_f) = (x_\infty - z, y_f) + (z, z)$. Now $x_\infty - z = y_\infty + n$ for some $y_\infty \in (0, 1]$ and $n \in \mathbb{Z}$. Hence

$$(x_\infty, x_f) = (y_\infty, y_f - n) + (n + z, n + z),$$

proving (2). (3) follows from (2). □

Let $\mathbf{U}$ be an affine variety defined over $\mathbb{Q}$. Choose any $\mathbb{Q}$-isomorphism α of $\mathbf{U}$ onto a Zariski closed $\mathbb{Q}$-subvariety $\mathbf{U}'$ of the N-dimensional affine space A^N for some N. Then the adele space $\mathbf{U}(\mathbb{A})$ corresponding to $\mathbf{U}$ is defined to be the restricted topological product of $\mathbf{U}(\mathbb{Q}_p)$'s with respect to $\alpha^{-1}(\mathbf{U}' \cap \mathbb{Z}_p^N)$'s. Note that this definition of $\mathbf{U}(\mathbb{A})$ does not depend on the choice of a $\mathbb{Q}$-isomorphism α. The set $\mathbf{U}(\mathbb{Q})$ imbeds into $\mathbf{U}(\mathbb{A})$ as a discrete subset.

The adele space $\mathbf{U}(\mathbb{A})$ for a general variety $\mathbf{U}$ is then defined using the open coverings of $\mathbf{U}$ by affine subsets.

2.3. $\mathbf{G}(\mathbb{Q})$ is a lattice in $\mathbf{G}(\mathbb{A})$ if $\mathbf{G}$ admits no non-trivial $\mathbb{Q}$-character. — Let $\mathbf{G} \subset \mathrm{GL}_n$ be a connected $\mathbb{Q}$-group. For a commutative ring J, $\mathrm{GL}_n(J)$ is defined to be the matrices with entries in J and with determinant being a unit in J. We set

$$\mathbf{G}(J) = \mathbf{G} \cap \mathrm{GL}_n(J).$$

Note that $\mathbf{G}(\mathbb{Z}_p)$ is a compact open subgroup of $\mathbf{G}(\mathbb{Q}_p)$ for each finite p. We claim that the adele space $\mathbf{G}(\mathbb{A})$ associated to $\mathbf{G}$ coincides with the restricted topological product of $\mathbf{G}(\mathbb{R})$ and $\mathbf{G}(\mathbb{Q}_p)$'s with respect to $\mathbf{G}(\mathbb{Z}_p)$'s for $p \in R_f$. To see this, we use the restriction, say α, to $\mathbf{G}$ of the map $\mathrm{GL}_n \to A^{n^2+1}$ given by $g \mapsto (g, \det(g)^{-1})$, where A^{n^2+1} is the n^2+1-dimensional affine space. Since $\alpha^{-1}(\alpha(\mathbf{G}) \cap \mathbb{Z}_p^{n^2+1}) = \mathbf{G}(\mathbb{Z}_p)$, this shows that $\mathbf{G}(\mathbb{A})$ defined at the end of the previous subsection is equal to $\prod_{p\in R}(\mathbf{G}(\mathbb{Q}_p) : \mathbf{G}(\mathbb{Z}_p))$.

We note that $\mathbf{G}(\mathbb{A})$, with the component-wise group operation, is a second countable locally compact group with a (left-invariant) Haar measure $\mu := \otimes^* \mu_p$ where μ_p is a (left-invariant) Haar measure on $\mathbf{G}(\mathbb{Q}_p)$ with $\mu_p(\mathbf{G}(\mathbb{Z}_p)) = 1$ for each finite $p \in R_f$.

Theorem 2.4. [**7**] *If $\mathbf{G}$ admits no non-trivial $\mathbb{Q}$-character, then $\mathbf{G}(\mathbb{Q})$ is a lattice in $\mathbf{G}(\mathbb{A})$.*

We give an outline of the proof of this theorem for $\mathbf{G} = \mathrm{SL}_n$. We often write an element of $\mathbf{G}(\mathbb{A})$ as $(g_\infty, g_f) \in \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{A}_f)$, where $\mathbf{G}(\mathbb{A}_f)$ denotes the subgroup of finite adeles, i.e., with the trivial component at ∞.

Theorem 2.5. *Let $\mathbf{G} = \mathrm{SL}_n$.*

(1) *$\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{A}_f)$.*
(2) *$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}) = \Sigma_0 \times \prod_{p\in R_f} \mathbf{G}(\mathbb{Z}_p)$ where $\Sigma_0 = \mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R})$.*
(3) *$\mathbf{G}(\mathbb{Q})$ is a lattice in $\mathbf{G}(\mathbb{A})$.*

Proof. For $1 \le i, j \le n$, denote by $U_{ij}(\mathbb{Q}_p)$ (resp. $U_{ij}(\mathbb{Q})$) the unipotent one parameter subgroup $I_n + \mathbb{Q}_p E_{i,j}$ (resp. $I_n + \mathbb{Q}E_{i,j}$) where E_{ij} is the matrix whose only non-zero entry is 1 at the (i, j)-entry. Since $\mathbb{Q}$ is dense in $\mathbb{A}_f$ and $U_{ij}(\mathbb{Q}_p)$'s generate $\mathrm{SL}_n(\mathbb{Q}_p)$, $\mathrm{SL}_n(\mathbb{Q}_p)$ is contained in the closure of $\mathrm{SL}_n(\mathbb{Q})$ in $\mathrm{SL}_n(\mathbb{A}_f)$, and hence any finite product of $\mathrm{SL}_n(\mathbb{Q}_p)$'s is contained in the closure of $\mathrm{SL}_n(\mathbb{Q})$ in $\mathrm{SL}_n(\mathbb{A}_f)$. Since they form a dense subset in $\mathrm{SL}_n(\mathbb{A}_f)$, this proves (1). By (1), we have

$$\mathrm{SL}_n(\mathbb{A}_f) = \mathrm{SL}_n(\mathbb{Q}) \prod_{p\in R_f} \mathrm{SL}_n(\mathbb{Z}_p).$$

It is easy to deduce (2) and (3) from this, using the facts that $\mathrm{SL}_n(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Q}) \cap \prod_{p\in R_f} \mathrm{SL}_n(\mathbb{Z}_p)$ and that $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$. □

3. General strategy on orbital counting

We now explain a general strategy for the orbital counting. This, at least in an explicit form, was first described and used in the work of Duke-Rudnick-Sarnak [**21**].

Let G be a locally compact second countable group and $L < G$ a closed subgroup. Let $\Gamma \subset G$ be a lattice such that the intersection $L \cap \Gamma$ is a lattice in L. This in particular implies that both G and L are unimodular, and that the orbit $x_0\Gamma$ is a discrete subset of $L\backslash G$ for $x_0 = [L]$. For a given sequence of growing compact subsets $B_n \subset L\backslash G$, we would like to understand the asymptotic behavior of $\#B_n \cap x_0\Gamma$ as $n \to \infty$. Heuristics suggest that

$$\#B_n \cap x_0\Gamma \sim \nu(B_n)$$

where ν is a measure on $L\backslash G$ determined as follows:

Notation 3.1. Let μ_G and μ_L be the Haar measures on G and L such that $\mu_G(\Gamma\backslash G) = 1 = \mu_L(\Gamma\cap L\backslash L)$. There exists the unique G-invariant measure ν on $L\backslash G$ which is compatible with μ_G and μ_L, in the sense that for any $\psi \in C_c(G)$,

$$\int_G \psi \, d\mu_G = \int_{[g]\in L\backslash G} \int_{h\in L} \psi(hg) d\mu_L(h) d\nu([g]).$$

Definition 3.2 (Counting function). Define the following counting function on $\Gamma\backslash G$:

$$F_n(g) = \sum_{\gamma\in\Gamma\cap L\backslash\Gamma} \chi_{B_n}(x_0\gamma g).$$

Noting that

$$F_n(e) = \#B_n \cap x_0\Gamma$$

we will present two conditions which guarantee that

$$F_n(e) \sim \nu(B_n) \quad \text{as } n \to \infty.$$

We denote by $\mathcal{P}(\Gamma\backslash G)$ the space of Borel probability measures on $\Gamma\backslash G$ and by $C_c(\Gamma\backslash G)$ the space of continuous functions on $\Gamma\backslash G$ with compact support. For a subgroup $K < G$, $C_c(\Gamma\backslash G)^K$ denotes a subset of $C_c(\Gamma\backslash G)$ consisting of right K-invariant functions.

Since $\Gamma\backslash\Gamma L = (\Gamma \cap L)\backslash L$ is a closed orbit in $\Gamma\backslash G$, we may consider μ_L as a probability measure in $\Gamma\backslash G$ supported in $\Gamma\backslash\Gamma L$.

Definition 3.3. (1) For $\mu \in \mathcal{P}(\Gamma\backslash G)$ and $g \in G$, we denote by $g.\mu$ the translation of μ by g, i.e.,

$$g.\mu_L(E) := \mu_L(Eg^{-1})$$

for any Borel subset E of $\Gamma\backslash G$.

(2) For a subset $\mathcal{F} \subset C_c(\Gamma\backslash G)$ and a sequence $\nu_i \in \mathcal{P}(\Gamma\backslash G)$, we say that ν_i weakly converges to μ, as $i \to \infty$, relative to $\mathcal{F}$ if for all $\psi \in \mathcal{F}$,

$$\lim_{i\to\infty} \nu_i(\psi) = \mu(\psi).$$

(3) For a sequence $g_i \in L\backslash G$, the translate $\Gamma\backslash\Gamma L g_i$ is said to become equidistributed in $\Gamma\backslash G$, as $i \to \infty$, relative to the family $\mathcal{F}$ if the sequence $(g_i).\mu_L$ weakly converges to μ_G relative to $\mathcal{F}$.
(4) If $\mathcal{F} = C_c(\Gamma\backslash G)$, we omit the reference to $\mathcal{F}$ in (2) and (3).

Definition 3.4. Let K be a compact subgroup of G. A family $\{B_n\}$ of K-invariant compact subsets of $L\backslash G$ is called K-well-rounded if there exists $c > 0$ such that for every $\epsilon > 0$, there exists a neighborhood U_ϵ of e in G satisfying

$$\nu(B_n U_\epsilon K - \cap_{g\in U_\epsilon K} B_n g) < c \cdot \epsilon \cdot \nu(B_n)$$

for all large n. For $K = \{e\}$, we simply say that B_n is well-rounded.

Proposition 3.5. *Let K be a compact subgroup of G and $\{B_n \subset L\backslash G\}$ a sequence of K-invariant compact subsets with $\nu(B_n) \to \infty$ as $n \to \infty$. Assume that the following hold:*
(1) *For any sequence $g_i \to \infty$ in $L\backslash G$, the translate $\Gamma\backslash\Gamma L g_i$ becomes equidistributed in $\Gamma\backslash G$ relative to the family $C_c(\Gamma\backslash G)^K$;*
(2) *The sequence $\{B_n\}$ is K-well-rounded.*

Then as $n \to \infty$

$$\# x_0\Gamma \cap B_n \sim \nu(B_n).$$

Proof. The proof consists of two steps. In the first step, we show that the condition (1) implies the weak-convergence of $\frac{1}{\nu(B_n)} F_n$, i.e., for all $\psi \in C_c(\Gamma\backslash G)^K$,

$$\lim_{n\to\infty} \frac{1}{\nu(B_n)} \int_{\Gamma\backslash G} F_n(x)\,\psi(x)\,d\mu_G(x) \quad = \quad \int_{\Gamma\backslash G} \psi(x)\,d\mu_G(x)\,. \tag{3.6}$$

Observe that

$$\begin{aligned}
\int_{\Gamma\backslash G} F_n\psi\,d\mu_G &= \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma\cap L\backslash\Gamma}\chi_{B_n}(x_0\gamma g)\psi(g)\right)d\mu_G(g)\\
&= \int_{\Gamma\cap L\backslash G}\chi_{B_n}(x_0 g)\psi(g)\,d\mu_G(g)\\
&= \int_{L\backslash G}\int_{\Gamma\cap L\backslash L}\chi_{B_n}(x_0 g)\,\psi(hg)\,d\mu_L(h)d\nu(g)\\
&= \int_{x_0 g\in B_n}\left(\int_{h\in\Gamma\backslash\Gamma L}\psi(hg)d\mu_L(h)\right)d\nu(g)\\
&= \int_{x_0 g\in B_n}\left(\int_{\Gamma\backslash G}\psi(x)\,d(g.\mu_L)(x)\right)d\nu(g)
\end{aligned}$$

By (1), for any $\epsilon > 0$, there exists a compact subset $C_\epsilon \subset L\backslash G$ such that

$$\sup_{x_0 g\notin C_\epsilon} |\int \psi d(g.\mu_L) - \int \psi d\mu_G| \le \epsilon.$$

Hence

$$|\int_{x_0 g\in B_n}\left(\int_{\Gamma\backslash G}\psi(x)\,d(g.\mu_L)(x)\right)d\nu(g) - \nu(B_n)\int\psi d\mu_G| \le 2\|\psi\|_\infty \nu(C_\epsilon\cap B_n) + \epsilon\nu(B_n).$$

Since $\nu(B_n) \to \infty$, we deduce that

$$\limsup_n \left| \frac{1}{\nu(B_n)} \int_{x_0 g \in B_n} \left(\int_{\Gamma\backslash G} \psi(x)\, d(g.\mu_L)(x) \right) d\nu(g) - \int \psi d\mu_G \right| \le \epsilon.$$

Hence (3.6) follows from (1).

In order to deduce the pointwise convergence from the weak convergence, we will now use the assumption that $\{B_n\}$ is K-well rounded. Fix $\epsilon > 0$ and $U_\epsilon \subset G$ be a symmetric open neighborhood of e in G as in the definition of the K-well-roundedness of B_n. Define $F^+_{n,\epsilon}$ and $F^-_{n,\epsilon}$ similarly to F_n but using $B^+_{n,\epsilon} := B_n U_\epsilon K$ and $B^-_{n,\epsilon} := \cap_{u \in U_\epsilon K} B_n u$ respectively, in place of B_n.

It is easy to check that for all $g \in U_\epsilon K$,

$$F^-_{n,\epsilon}(g) \le F_n(e) \le F^+_{n,\epsilon}(g). \tag{3.7}$$

Choose a K-invariant non-negative continuous function ψ_ϵ on $\Gamma\backslash G$ with support in $\Gamma\backslash\Gamma U_\epsilon K$ and with the integral $\int_{\Gamma\backslash G} \psi_\epsilon d\mu_G$ one.

By integrating (3.7) against ψ_ϵ, we have

$$\langle F^-_{n,\epsilon}, \psi_\epsilon \rangle \le F_n(e) \le \langle F^+_{n,\epsilon}, \psi_\epsilon \rangle.$$

Applying (3.6) to $F^\pm_{n,\epsilon}$, which we may since $\nu(B^\pm_{n,\epsilon}) \to \infty$, we have

$$\langle F^\pm_{n,\epsilon}, \psi_\epsilon \rangle \sim_n \nu(B^\pm_{n,\epsilon}).$$

Therefore there are constants $c_1, c_2 > 0$ such that for any $\epsilon > 0$,

$$\limsup_n \frac{F_n(e)}{\nu(B_n)} \le (1 + c_1\epsilon) \cdot \limsup_n \frac{\nu(B^+_{n,\epsilon})}{\nu(B_n)} \le (1 + c_1\epsilon)(1 + c_2\epsilon).$$

Similarly,

$$(1 - c_1\epsilon)(1 - c_2\epsilon) \le \liminf_n \frac{F_n(e)}{\nu(B_n)}.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{n\to\infty} \frac{F_n(e)}{\nu(B_n)} = 1.$$

□

Remark 3.8. The above proposition was considered only for $K = \{e\}$ in [**21**] (also in [**26**]). In applications where G and L are (the identity components of) real Lie groups, this is usually sufficient. However when G and L are the adelic groups associated to non-simply connected semisimple $\mathbb{Q}$-groups, the equidistribution in Prop. 3.5 (1) does not usually hold for all of $C_c(\Gamma\backslash G)$ (cf. Theorem 4.3). Hence it is necessary to consider the above modification by introducing compact subgroups K.

Let $\mu_n \in \mathcal{P}(\Gamma\backslash G)$ denote the average of measures $x.\mu_L$, $x \in B_n$: for $\psi \in C_c(\Gamma\backslash G)$,

$$\mu_n(\psi) := \frac{1}{\nu(B_n)} \int_{x \in B_n} (x.\mu_L)(\psi)\, d\nu(x) = \frac{1}{\nu(B_n)} \int_{x \in B_n} \int_{\Gamma\cap L\backslash L} \psi(hx)\, d\mu_L(h)\, d\nu(x).$$

The proof of the above proposition yields the following stronger version:

Proposition 3.9. *Let K be a compact subgroup of G and $\{B_n \subset L\backslash G\}$ a sequence of K-invariant compact subsets with $\nu(B_n) \to \infty$ as $n \to \infty$. Suppose:*

1. *μ_n weakly converges to μ_G relative to $C_c(\Gamma\backslash G)^K$;*
2. *The sequence $\{B_n\}$ is K-well-rounded.*

Then as $n \to \infty$,

$$\# x_0 \Gamma \cap B_n \sim_n \nu(B_n).$$

4. Equidistribution of semisimple periods via unipotent flows

Let $\mathbf{G} \subset \mathrm{GL}_n$ be a connected semisimple algebraic $\mathbb{Q}$-group and $\mathbf{L}$ a connected semisimple $\mathbb{Q}$-subgroup of $\mathbf{G}$. Note that $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ and $\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})$ have finite volumes by Theorem 2.4. We are interested in the asymptotic behavior of the translate

$$\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g_i \subset \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$$

as $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$.

We hope that the translate $\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g_i$ becomes equidistributed in the space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ as $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$. One obvious obstruction is the existence of a proper $\mathbb{Q}$-subgroup $\mathbf{M}$ of $\mathbf{G}$ which contains $\mathbf{L}$ properly, since the sequence $\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g_i$ then remains entirely inside the closed subset $\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})$ for $g_i \in \mathbf{M}(\mathbb{A})$, and hence the desired equidistribution cannot happen for those sequences $g_i \in \mathbf{M}(\mathbb{A})$.

In the case when both $\mathbf{G}$ and $\mathbf{L}$ are simply connected, this is the only obstruction. In the rest of this section, we assume that $\mathbf{L}$ is a maximal connected $\mathbb{Q}$-subgroup of $\mathbf{G}$, unless mentioned otherwise. We closely follow the exposition in [**35**] to which we refer for details.

Theorem 4.1. *Suppose that both $\mathbf{G}$ and $\mathbf{L}$ are simply connected. Then $\mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g_i$ becomes equidistributed in $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ for any sequence $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$.*

In the general case, when $\mathbf{G}$ and $\mathbf{L}$ are not necessarily simply connected, the above theorem does not hold, because there are many finite index subgroups of $\mathbf{G}(\mathbb{A})$ which contain $\mathbf{G}(\mathbb{Q})$ as a lattice and the entire dynamics may happen only in these smaller pieces of $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$.

To overcome this issue, we consider a simply connected covering $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ over $\mathbb{Q}$. The map π induces a map from $\tilde{\mathbf{G}}(\mathbb{A})$ to $\mathbf{G}(\mathbb{A})$, which we again denote by π by abuse of notation.

Lemma 4.2. *For any compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$, the product*

$$G_W := \mathbf{G}(\mathbb{Q})\pi(\tilde{\mathbf{G}}(\mathbb{A}))W$$

is a normal subgroup of $\mathbf{G}(\mathbb{A})$ with finite index.

Theorem 4.3. [**35**] *Fix a compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$. Then for any $g_i \to \infty$ in $\mathbf{L}(\mathbb{A})\cap G_W\backslash G_W$, the translate $\mathbf{L}(\mathbb{Q})\backslash(\mathbf{L}(\mathbb{A})\cap G_W)g_i$ becomes equidistributed in $\mathbf{G}(\mathbb{Q})\backslash G_W$ relative to the family $C_c(\mathbf{G}(\mathbb{Q})\backslash G_W)^W$.*

Analogous statement for $\mathbf{L}$ being a maximal $\mathbb{Q}$-anisotropic torus was proved in [**64**] for $\mathbf{G} = \mathrm{PGL}_2$ and in [**23**] for $\mathbf{G} = \mathrm{PGL}_3$.

In the rest of this section, we will outline the proof of Theorem 4.1; so we assume that both $\mathbf{G}$ and $\mathbf{L}$ are simply connected. The first step is to reduce the equidistribution problem in the homogeneous spaces of an adele group to the S-algebraic setting, using the strong approximation theorem.

It is convenient to define the following:

Definition 4.4.
(1) For a semisimple $\mathbb{Q}$-subgroup $\mathbf{L}$ of $\mathbf{G}$, an element $p \in R$ is called isotropic for $\mathbf{L}$ if $\mathbf{N}(\mathbb{Q}_\mathbf{p})$ is non-compact for any non-trivial normal $\mathbb{Q}$-subgroup $\mathbf{N}$ of $\mathbf{L}$.
(2) For a semisimple $\mathbb{Q}$-subgroup $\mathbf{L}$ of $\mathbf{G}$, an element $p \in R$ is called strongly isotropic for $\mathbf{L}$ if $\mathbf{N}(\mathbb{Q}_\mathbf{p})$ is non-compact for any non-trivial normal $\mathbb{Q}_p$-subgroup $\mathbf{N}$ of $\mathbf{L}$.

(3) A finite subset S of R is called strongly isotropic (resp. isotropic) for $\mathbf{L}$ if S contains a strongly isotropic (resp. isotropic) p for $\mathbf{L}$.

We fix a finite subset $S \subset R$ containing ∞ in the rest of this section. We set

$$\mathbf{G}_S := \prod_{p \in S} \mathbf{G}(\mathbb{Q}_p).$$

Denoting by $\mathbf{G}(\mathbb{A}_S)$ the subgroup of $\mathbf{G}(\mathbb{A})$ with trivial S-components, the group $\mathbf{G}(\mathbb{A})$ can be naturally identified with the product $\mathbf{G}_S \times \mathbf{G}(\mathbb{A}_S)$.

THEOREM 4.5 (Strong approximation property). *Let S be $\mathbf{G}$-isotropic. Then for any compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$,*

$$\mathbf{G}(\mathbb{Q})W_S = \mathbf{G}(\mathbb{A}_S).$$

See [**51**, 7.4].

Hence any element $g \in \mathbf{G}(\mathbb{A})$ can be written as

$$g = (\gamma_g, \gamma_g)(g_S, w)$$

where $\gamma_g \in \mathbf{G}(\mathbb{Q})$, $g_S \in \mathbf{G}_S$ and $w \in W_S$. Note that g_S is determined uniquely up to the left multiplication by an element of $\mathbf{G}(\mathbb{Q}) \cap W_S$.

DEFINITION 4.6. A subgroup Γ of $\mathbf{G}(\mathbb{Q})$ is called an S-congruence subgroup if $\Gamma = \mathbf{G}(\mathbb{Q}) \cap W_S$ for some compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$.

Note that an S-congruence subgroup is a lattice in $\mathbf{G}_S$, embedded diagonally.

EXAMPLE 4.7. If $S = \{\infty, p\}$ and $W := \prod_{q \neq p} \mathrm{SL}_n(\mathbb{Z}_q)$, then the diagonal embedding of $\Gamma = \mathrm{SL}_n(\mathbb{Q}) \cap W = \mathrm{SL}_n(\mathbb{Z}[1/p])$ is a lattice in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$.

PROPOSITION 4.8. *Let S be isotropic for both $\mathbf{G}$ and $\mathbf{L}$, and W_S a compact open subgroup of $\mathbf{G}(\mathbb{A}_S)$. Set $\Gamma := \mathbf{G}(\mathbb{Q}) \cap W_S$.*

(1) *The map $g \mapsto g_S$ induces a $\mathbf{G}_S$-equivariant topological isomorphism*

$$\Phi : \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/W_S \to \Gamma\backslash\mathbf{G}_S.$$

(2) *For any $g = (\gamma_g, \gamma_g)(g_S, w) \in \mathbf{G}(\mathbb{A})$, the map Φ, via the restriction, induces the topological isomorphism*

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})\mathbf{L}(\mathbb{A})gW_S/W_S \simeq \Gamma\backslash\Gamma\gamma_g^{-1}\mathbf{L}_S\gamma_g g_S.$$

PROOF. Since the map $g \mapsto (g_S, w)$ induces a $\mathbf{G}_S$-equivariant homeomorphism between $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ and $\Gamma\backslash(\mathbf{G}_S \times W_S)$, (1) follows. The strong approximation theorem 4.5 applied to $\mathbf{L}$ implies

$$\mathbf{L}(\mathbb{A}) = \mathbf{L}(\mathbb{Q})\mathbf{L}_S(gW_Sg^{-1} \cap \mathbf{L}(\mathbb{A}_S)).$$

Hence any $x \in \mathbf{L}(\mathbb{A})$ can be written, modulo $\mathbf{L}(\mathbb{Q})$, as $(x_S, gw'g^{-1})$ for some $w' \in W_S$ and $x_S \in \mathbf{L}_S$. Now

$$xg = (\gamma_g, \gamma_g)(\gamma_g^{-1}x_S\gamma_g g_S, ww').$$

Hence $\Phi[xg]$ is represented by $\gamma_g^{-1}x_S\gamma_g g_S$ in $\Gamma\backslash\mathbf{G}_S$. □

The above proposition implies the following:

LEMMA 4.9 (Basic Lemma). *Let S be isotropic both for $\mathbf{G}$ and $\mathbf{L}$. For a sequence $g \in \mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$ going to infinity, the following are equivalent:*

- *The translate $\tilde{Y}_g := \mathbf{L}(\mathbb{Q})\backslash\mathbf{L}(\mathbb{A})g$ becomes equidistributed in $\tilde{X} := \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$;*

- *For any compact open subgroup W_S of $\mathbf{G}(\mathbb{A}_S)$ and the corresponding S-congruence subgroup $\Gamma = \mathbf{G}(\mathbb{Q}) \cap W_S$, the translate $Y_g := \Gamma\backslash\Gamma\gamma_g^{-1}\mathbf{L}_S\gamma_g g_S$ becomes equidistributed in $X := \Gamma\backslash\mathbf{G}_S$.*

Proof. Let $\tilde{\mu}_g$, $\tilde{\mu}$, μ_g and μ be the invariant probability measures on $\tilde{Y}_g$, $\tilde{X}$, Y_g and X respectively. Note that $\bigcup_{W_S} C_c(\tilde{X})^{W_S}$, where the union is taken over all open compact subgroups W_S of $\mathbf{G}(\mathbb{A}_S)$, is dense in $C_c(\tilde{X})$, and that any function $\tilde{f} \in C_c(\tilde{X})^{W_S}$ corresponds to a unique function $f \in C_c((\mathbf{G}(\mathbb{Q})\cap W_S)\backslash\mathbf{G}_S)$, and vice versa, by Proposition 4.8. Moreover

$$\tilde{\mu}_g(\tilde{f}) = \mu_g(f) \text{ and } \tilde{\mu}(\tilde{f}) = \mu(f).$$

Therefore the claim follows.

□

In the following theorem, let $\mathbf{G}$ be a connected simply connected semisimple $\mathbb{Q}$-group and $\{\mathbf{L}_i\}$ a sequence of simply connected semisimple $\mathbb{Q}$-groups of $\mathbf{G}$. Suppose that S is *strongly* isotropic for all $\mathbf{L}_i$. Let Γ be an S-congruence subgroup of $\mathbf{G}(\mathbb{Q})$, and denote by μ_i the invariant probability measure supported on the closed orbit $\Gamma\backslash\Gamma\mathbf{L}_{i,S}$.

The following theorem is proved in [**35**], generalizing the works of Mozes-Shah [**46**], and of Eskin-Mozes-Shah ([**27**], [**25**]).

Theorem 4.10. *Let $\{g_i \in \mathbf{G}_S\}$ be given.*

(1) *If the centralizer of each $\mathbf{L}_i$ is $\mathbb{Q}$-anisotropic, then $(g_i).\mu_i$ does not escape to infinity, that is, for any $\epsilon > 0$, there is a compact subset $\Omega \subset \Gamma\backslash\mathbf{G}_S$ such that*

$$(g_i).\mu_i(\Omega) > 1 - \epsilon \quad \textit{for all large } i.$$

(2) *If $\nu \in \mathcal{P}(\Gamma\backslash\mathbf{G}_S)$ is a weak-limit of $(g_i).\mu_i$, then the following hold:*
- $$\mathrm{supp}(\nu) = \Gamma\backslash\Gamma g\Lambda(\nu)$$
  where $\Lambda(\nu) := \{x \in \mathbf{G}_S : x.\nu = \nu\}$, $g \in \mathbf{G}_S$ and $g\Lambda(\nu)g^{-1}$ is a finite index subgroup of $\mathbf{M}_S$ for some connected $\mathbb{Q}$-group $\mathbf{M}$ with no non-trivial $\mathbb{Q}$-character.
- *For some $\gamma_i \in \Gamma$,*
  $$\gamma_i\mathbf{L}_i\gamma_i^{-1} \subset \mathbf{M}$$
  and for some $h_i \in \mathbf{L}_S$, the sequence $\gamma_i h_i g_i$ converges to g.

The proof of this theorem is based on the theory of unipotent flows on homogeneous spaces. To see which unipotent flows we use, pick $p \in S$ such that each $\mathbf{L}_i$ has no anisotropic $\mathbb{Q}_p$-factor. Since S is finite, we may assume the existence of such p, by passing to a subsequence if necessary. Then $\mathbf{L}_i(\mathbb{Q}_p)$ is generated by unipotent one-parameter subgroups in it and acts ergodically on each $\Gamma\backslash\Gamma\mathbf{L}_{i,S}$. From this, we deduce that there exists a one-parameter unipotent subgroup U_i in $\mathbf{L}_i(\mathbb{Q}_p)$ which acts ergodically on $\Gamma\backslash\Gamma\mathbf{L}_{i,S}$. Then $U_i' := g_i^{-1}U_i g_i$ acts ergodically on $\Gamma\backslash\Gamma\mathbf{L}_{i,S}g_i$ and the measure $(g_i).\mu_i$ is a U_i'-invariant ergodic measure. We then need to understand the limiting behavior of invariant ergodic measures on $\Gamma\backslash\mathbf{G}_S$ under unipotent one parameter subgroups.

The rest of the proof is then based on the generalization to the S-arithmetic setting of theorems of Mozes-Shah on limits of ergodic measures invariant under unipotent flows [**46**] and of Dani-Margulis [**16**] on the behavior of unipotent flows near cusps. Main ingredients of the proof are the measure classification theorem of Ratner [**52**], generalized in the S-arithmetic setting by Ratner [**53**] and independently by Margulis-Tomanov [**43**], and the linearization methods developed by Dani-Margulis [**17**].

Note that $g \to \infty$ in $\mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})$ if and only if $(g_S, \gamma_g) \to \infty$ in $\mathbf{L}_S\backslash\mathbf{G}_S \times \mathbf{L}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})/\Gamma$. Therefore, using the Basic lemma 4.9, Theorem 4.1 follows from the following corollary of Theorem 4.10:

Corollary 4.11. *Suppose that* $\mathbf{L}$ *is a maximal connected* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$*. For* $(g_i, \delta_i) \to \infty$ *in* $\mathbf{L}_S\backslash\mathbf{G}_S \times \Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{L}(\mathbb{Q})$ *the sequence* $\Gamma\backslash\Gamma\delta_i\mathbf{L}_S g_i$ *becomes equidistributed in* $\Gamma\backslash\mathbf{G}_S$*.*

5. Well-rounded sequence and Counting rational points

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group and $\mathbf{L}$ a semisimple maximal connected $\mathbb{Q}$-subgroup of $\mathbf{G}$. Let $\mathbf{U} = \mathbf{L}\backslash\mathbf{G}$ and $u_0 = [\mathbf{L}]$.

Theorem 4.3 yields the following corollary by Proposition 3.5:

Corollary 5.1. *If* $\{B_T \subset \mathbf{L}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})\}$ *is a family of compact subsets which is* W*-well-rounded for some compact open subgroup* W *of* $\mathbf{G}(\mathbb{A}_f)$*, then*

$$\# u_0\mathbf{G}(\mathbb{Q}) \cap B_T \sim \nu(B_T \cap u_0 G_W)$$

provided $\nu(B_T \cap u_0 G_W) \to \infty$*, where* ν *is the invariant measure on* $u_0 G_W$ *which is compatible with invariant probability measures on* $\mathbf{G}(\mathbb{Q})\backslash G_W$ *and* $\mathbf{L}(\mathbb{Q})\backslash(G_W \cap \mathbf{L}(\mathbb{A}))$*.*

Proof. In order to apply Proposition 3.5, we set $G = G_W$, $L = G_W \cap \mathbf{L}(\mathbb{A})$ for $\mathbf{L} = \mathrm{Stab}_{\mathbf{G}}(u_0)$ and $\Gamma = \mathbf{G}(\mathbb{Q})$. By Theorem 4.3, the translate $\Gamma\backslash\Gamma L x$ becomes equidistributed in $\Gamma\backslash G$ relative to $C_c(\Gamma\backslash G)^W$, as $x \to \infty$ in $L\backslash G$. Hence the claim follows from Proposition 3.5. □

5.1. Rational points on projective varieties. Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group with a $\mathbb{Q}$-rational representation $\mathbf{G} \to \mathrm{GL}_{n+1}$. Let $\mathbf{U} := u_0\mathbf{G} \subset \mathbb{P}^n$ for some $u_0 \in \mathbb{P}^n(\mathbb{Q})$. Let $\mathbf{X} \subset \mathbb{P}^n$ denote the Zariski closure of $\mathbf{U}$, which is then a $\mathbf{G}$-equivariant compactification of $\mathbf{U}$. We assume that the stabilizer $\mathbf{L}$ in $\mathbf{G}$ of u_0 is connected and semisimple.

Let L be the line bundle of $\mathbf{X}$ given by the pull back of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Then L is very ample and $\mathbf{G}$-linearized. Let $\mathrm{s}_0, \cdots, \mathrm{s}_n$ be the global sections of L obtained by pulling back the coordinate functions x_i's.

Recall that the height function H_{L} on $\mathbf{X}(\mathbb{Q})$ is then given as follows: for $x \in \mathbf{X}(\mathbb{Q})$,

$$\mathrm{H}_{\mathrm{L}}(x) := \mathrm{H}_{\mathcal{O}_{\mathbb{P}^n}(1)}(x) = \sqrt{x_0^2 + \cdots + x_n^2} \tag{5.2}$$

where $(x_0, \cdots, x_n)$ is a primitive integral vector proportional to $(\mathrm{s}_0(x), \cdots, \mathrm{s}_n(x))$. In order to extend H_{L} to $\mathbf{U}(\mathbb{A})$, we assume that there is a $\mathbf{G}$-invariant global section s of L such that $\mathbf{U} = \{\mathrm{s} \neq 0\}$.

Definition 5.3. Define $\mathrm{H}_{\mathrm{L}} : \mathbf{U}(\mathbb{A}) \to \mathbb{R}_{>0}$ by

$$\mathrm{H}_{\mathrm{L}}(x) := \prod_{p \in R} \mathrm{H}_{\mathrm{L},p}(x_p) \quad \text{for } x = (x_p)$$

where

$$\mathrm{H}_{\mathrm{L},p}(x_p) = \begin{cases} \max_{0 \le i \le n} \left| \frac{\mathrm{s}_i(x_p)}{\mathrm{s}(x_p)} \right|_p & \text{for } p \text{ finite} \\ \frac{\sqrt{\sum_i \mathrm{s}_i(x_\infty)^2}}{|\,\mathrm{s}(x_\infty)|_\infty} & \text{for } p = \infty \end{cases}.$$

Observe that this definition of $\mathrm{H_L}$ agrees with the one in (5.2) on $\mathbf{U}(\mathbb{Q})$, using the product formula $\prod_{p\in R}|\,\mathrm{s}(x)|_p = 1$ for $x \in \mathbf{U}(\mathbb{Q})$.

Set

$$B_T := \{x \in \mathbf{U}(\mathbb{A}) : \mathrm{H_L}(x) \le T\};$$
$$W_{\mathrm{L}} := \{g \in \mathbf{G}(\mathbb{A}_f) : \mathrm{H_L}(xg) = \mathrm{H_L}(x) \quad \text{for all } x \in \mathbf{U}(\mathbb{A})\}.$$

Then W_{L} is an open compact subgroup of $\mathbf{G}(\mathbb{A}_f)$ under which B_T is invariant.

In view of Corollary 5.1, we would like to show that $B_T \cap u_0 G_{W_{\mathrm{L}}}$ is W_{L}-well rounded. So far we have completely ignored any geometry of $\mathbf{X}$; here is the place where it enters into the counting problem of rational points.

In the following theorem, we suppose that there are only finitely many $\mathbf{G}(\mathbb{A})$-orbits in $\mathbf{U}(\mathbb{A})$. This finiteness condition is equivalent to that $\mathbf{G}(\mathbb{Q}_p)$ acts transitively on $\mathbf{U}(\mathbb{Q}_p)$ for almost all p, as well as to that there are only finitely many $\mathbf{G}(\mathbb{Q})$-orbits in $\mathbf{U}(\mathbb{Q})$. This is always satisfied if $\mathbf{L}$ is simply connected. Borovoi [**35**] classified symmetric spaces $\mathbf{L}\backslash\mathbf{G}$ with this finiteness property when $\mathbf{G}$ is absolutely simple.

THEOREM 5.4. *For any $x_0 \in \mathbf{U}(\mathbb{A})$, we have*

(1) *the family $\{x_0 G_{W_{\mathrm{L}}} \cap B_T\}$ is W_{L}-well rounded;*
(2) *for some $a \in \mathbb{Q}_{>0}$ and $b \in \mathbb{Z}_{\ge 1}$ (explicitly given in terms of* $\mathrm{div}(\mathrm{s})$ *and the canonical class $\mathrm{K}_{\mathbf{X}}$),*
$$\mathrm{Vol}(x_0 G_{W_{\mathrm{L}}} \cap B_T) \asymp T^a \log T^{b-1}$$
($\asymp$ means the ratio of the two sides is between bounded constants uniformly for all $T > 1$).

To give some idea of the proof, consider the height zeta function on $x_0\mathbf{G}(\mathbb{A})$:

$$\eta(s) := \int_{x_0\mathbf{G}(\mathbb{A})} \mathrm{H_L}(x)^{-s} d\nu(x).$$

Understanding the analytic properties of η provides the asymptotic growth of $\nu(x_0\mathbf{G}(\mathbb{A}) \cap B_T)$ by Tauberian type argument. By the assumption on the finiteness, $x_0\mathbf{G}(\mathbb{Q}_p) = \mathbf{U}(\mathbb{Q}_p)$ for almost all p, and hence for some finite subset $S \subset R$,

$$\eta(s) = \prod_{p\in S} \int_{x_p\mathbf{G}(\mathbb{Q}_p)} \mathrm{H}_{\mathrm{L},p}^{-s}(y_p)\; d\nu_p(y_p) \cdot \prod_{p\notin S} \int_{\mathbf{U}(\mathbb{Q}_p)} \mathrm{H}_{\mathrm{L},p}^{-s}(y_p)\; d\nu_p(y_p)$$

where $\nu = \otimes^*\nu_p$ and $x = (x_p)$. Using the equivariant resolution of singularities and by passing to a finite field extension, we may assume that $\mathbf{X}$ is smooth with $\mathbf{X} \setminus \mathbf{U}$ a strict normal crossing divisor consisting of geometrically irreducible components:

$$\mathbf{X} \setminus \mathbf{U} = \cup_{\alpha\in\mathcal{A}} D_\alpha.$$

If ω denotes a nowhere zero differential form on $\mathbf{U}$ of top degree, we write

$$\mathrm{div}(\mathrm{s}) = \sum_{\alpha\in\mathcal{A}} m_\alpha D_\alpha \quad \text{and} \quad -\mathrm{div}(\omega) = \sum_{\alpha\in\mathcal{A}} n_\alpha D_\alpha$$

for $m_\alpha \in \mathbb{N}$ and $n_\alpha \in \mathbb{Z}$. Each of the local integral $\int_{\mathbf{U}(\mathbb{Q}_p)} \mathrm{H}_{\mathrm{L},p}^{-s}(y_p)\; d\nu_p(y_p)$ admits a formula analogous to Denef's formula for Igusa zeta function. And putting these together, one can regularize $\eta(s)$ by the Dedekind zeta function and obtain that $\eta(s)$ has a meromorphic continuation to the half plane $\Re(s) \ge a - \epsilon$ for some $\epsilon > 0$ with a unique pole at $s = a$ of order b, where

$$a = \max_{\alpha\in\mathcal{A}} \frac{n_\alpha}{m_\alpha} \quad \text{and} \quad b = \#\{\alpha \in \mathcal{A} : \frac{n_\alpha}{m_\alpha} = a\}.$$

This argument has been carried out by Chambert-Loir and Tschinkel [**11**]. We have that $a > 0$ (see [**5**]) and by Tauberian argument that

$$\nu(x_0\mathbf{G}(\mathbb{A}) \cap B_T) \sim c \cdot T^a (\log T)^{b-1}.$$

Note here that using the finiteness of $\mathbf{G}(\mathbb{A})$-orbits on $\mathbf{U}(\mathbb{A})$, the computation for the local integrals over $x_p\mathbf{G}(\mathbb{Q}_p)$ at almost all p becomes that over $\mathbf{X}(\mathbb{Q}_p)$ and hence a geometric problem. Without this assumption, one probably needs to use motivic integration.

Since $x_0\mathbf{G}(\mathbb{A})$ can be covered by finitely many translates of $x_0G_{W_{\mathrm{L}}}$, it is easy to deduce from here that $\nu(x_0G_{W_{\mathrm{L}}} \cap B_T) \asymp T^a(\log T)^{b-1}$, although it does not yield the asymptotic equality. The W_{L}-well-roundedness of the sequence $x_0G_{W_{\mathrm{L}}} \cap B_T$ does not immediately follow from this as well, but requires knowing a subtle Hölder property of local integral at ∞ (see Benoist-Oh [**3**], or Gorodnik-Nevo [**34**]).

Theorem 5.5. *Assume that*

(i) $\mathbf{L}$ *is a maximal connected* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$*;*
(ii) *there are only finitely many* $\mathbf{G}(\mathbb{A})$*-orbits in* $\mathbf{U}(\mathbb{A})$*.*

Then

(1) *for any* $u_0 \in \mathbf{U}(\mathbb{Q})$*,*

$$\#\{x \in u_0\mathbf{G}(\mathbb{Q}) : \mathrm{H}_{\mathrm{L}}(x) < T\} \sim \nu(B_T \cap u_0G_{W_{\mathrm{L}}})$$

where ν *is the invariant measure on* $u_0G_{W_{\mathrm{L}}}$ *which is compatible with invariant probability measures on* $\mathbf{G}(\mathbb{Q})\backslash G_{W_{\mathrm{L}}}$ *and* $\mathbf{L}(\mathbb{Q})\backslash(G_{W_{\mathrm{L}}} \cap \mathbf{L}(\mathbb{A}))$*.*

(2) *there exist* $a \in \mathbb{Q}_{>0}$ *and* $b \in \mathbb{N}$ *(explicitly given in terms of* $\mathrm{div}(s)$ *and the canonical class of* $\mathbf{X}$*) such that*

$$\#\{x \in \mathbf{U}(\mathbb{Q}) : \mathrm{H}_{\mathrm{L}}(x) < T\} \asymp T^a \log T^{b-1}.$$

Generalizing the work of De Concini and Procesi [**18**] on the construction of the wonderful compactification of symmetric varieties, Luna introduced in [**42**] the notion of a wonderful variety: a smooth connected projective $\mathbf{G}$-variety $\mathbf{X}$ is called *wonderful* of rank l if (1) $\mathbf{X}$ contains l irreducible $\mathbf{G}$-invariant divisors with strict normal crossings, (2) $\mathbf{G}$ has exactly 2^l-orbits in $\mathbf{X}$. In particular, a wonderful variety is of Fano type. For a $\mathbf{G}$-homogeneous variety $\mathbf{U}$, a wonderful variety $\mathbf{X}$ is called the wonderful compactification of $\mathbf{U}$ if it is a $\mathbf{G}$-equivariant compactification of $\mathbf{U}$. Using the work of Brion on the computation of $\mathrm{Pic}(\mathbf{X})$ and $\Lambda_{\mathrm{eff}}(\mathbf{X})$ etc., we can verify that $a = a_{\mathrm{L}}$ and $b = b_{\mathrm{L}}$ as predicted by Manin and the height function H_{L} associated to a very ample line bundle L of $\mathbf{X}$ arises as described in the beginning of (5.1) provided $\mathbf{L}$ has finite index in its normalizer. Therefore we deduce the following special case of Manin's conjecture.

Theorem 5.6. *Under the same assumption as in Theorem 5.5, let* $\mathbf{X}$ *be a wonderful variety. Then for any ample line bundle* L *of* $\mathbf{X}$ *over* $\mathbb{Q}$*,*

$$\#\{x \in \mathbf{U}(\mathbb{Q}) : \mathrm{H}_{\mathrm{L}}(x) < T\} \asymp T^{a_{\mathrm{L}}} \log T^{b_{\mathrm{L}}-1}$$

where a_{L} *and* b_{L} *are as in Manin's conjecture. Moreover if* $\mathbf{G}$ *is simply connected, there exists* $c > 0$ *such that*

$$\#\{x \in \mathbf{U}(\mathbb{Q}) : \mathrm{H}_{\mathrm{L}}(x) < T\} \sim c \cdot T^{a_{\mathrm{L}}} \log T^{b_{\mathrm{L}}-1}.$$

The above theorem applies to the wonderful compactification of a connected adjoint semisimple algebraic group $\mathbf{G}$, since $\mathbf{G}$ can be identified with $\Delta(\mathbf{G})\backslash\mathbf{G} \times \mathbf{G}$ where $\Delta(\mathbf{G})$ is the diagonal embedding of $\mathbf{G}$ into $\mathbf{G} \times \mathbf{G}$. This case was previously obtained in [**58**] with rate of convergence and an alternative approach was given in [**33**] (see [**61**] for the comparison of two methods).

5.2. Rational points on affine varieties. The main difference of the counting problem between an affine variety $\mathbf{V}$ and a Zariski open subset $\mathbf{U}$ of a projective variety $\mathbf{X}$ lies in the way of defining a height function. Recall that a height function on $\mathbf{U}$ is obtained by pulling back the height function on the projective space into which $\mathbf{X}$ is embedded. For an affine variety $\mathbf{V}$ in an affine n-space, one could also try to embed it into a projective space and use the height function there. However it is natural to ask if the following definition of a height works: for $x = (x_p) \in \mathbf{V}(\mathbb{A})$,

$$\mathrm{H}(x) := \prod_{p\in R} \|x_p\|_p,$$

where $\|\cdot\|_\infty$ is the Euclidean norm on $\mathbb{R}^n$, and $\|\cdot\|_p$ is the p-adic maximum norm on $\mathbb{Q}_p^n$ for each finite p.

For a general affine variety $\mathbf{V}$ (for instance, for the affine n-space), this may not be well-defined. We discuss the case of homogeneous affine varieties in the following.

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group with a $\mathbb{Q}$-rational representation $\mathbf{G} \to \mathrm{GL}_n$. Fix a non-zero vector $v_0 \in \mathbb{Q}^n$ such that the orbit $\mathbf{V} = v_0\mathbf{G}$ is an affine $\mathbb{Q}$-subvariety. We assume that the stabilizer $\mathbf{L}$ in $\mathbf{G}$ of v_0 is a semisimple maximal connected $\mathbb{Q}$-subgroup of $\mathbf{G}$.

Lemma 5.7. *For almost all p,*

$$\delta_p := \min_{x\in\mathbf{V}(\mathbb{Q})} \mathrm{H}_p(x) := \|x\|_p \geq 1.$$

Proof. It is well known that there exists a $\mathbf{G}$-invariant non-zero homogeneous polynomial f with integral coefficients, that is, $\mathbf{V} \subset \{f = r\}$ for some $r \in \mathbb{Q}\setminus\{0\}$. Now for any p coprime to r as well as to the coefficients of f, we claim that $\delta_p \geq 1$. Suppose not; for some $x \in \mathbf{V}(\mathbb{Q})$, p divides each coordinate of x. Write $x = p^k x'$ where $k \geq 1$ and the denominator of any coordinate of x' is divisible by p. Now if d is the degree of f, $f(x) = p^{kd} f(x') = r$ and $|f(x')|_p \leq 1$. Hence $|r|_p \leq p^{-kd}$, yielding contradiction. $\square$

We write an element of $\mathbf{V}(\mathbb{Q})$ as $\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$ where $x_0, \cdots, x_n \in \mathbb{Z}$, $x_0 > 0$ and g.c.d$(x_0, \cdots, x_n) = 1$.

Lemma 5.8. *For $x \in \mathbf{V}(\mathbb{Q})$,*

$$\mathrm{H}(x) \asymp \sqrt{x_0^2 + \cdots + x_n^2},$$

in the sense that the ratio is between two bounded constants uniformly for all $x \in \mathbf{V}(\mathbb{Q})$.

Proof. By the product formula,

$$\begin{aligned}\mathrm{H}(x) &= \prod_p |x_0|_p \cdot \mathrm{H}(x)\\ &\leq \prod_p |x_0|_p \sqrt{\sum_{i=1}^n \frac{x_i^2}{x_0^2} + 1} \cdot \prod_{p\in R_f} \max_{1\leq i\leq n} \{\frac{|x_i|_p}{|x_0|_p}, 1\}\\ &\leq \sqrt{\sum_{0\leq i\leq n} x_i^2} \cdot \prod_{p\in R_f} \max_{0\leq i\leq n} |x_i|_p = \sqrt{\sum_{0\leq i\leq n} x_i^2}.\end{aligned}$$

By Lemma 5.7 and its proof,

$$\|x\|_p = \max_i \{ \frac{|x_i|_p}{|x_0|_p}, \delta_p \}$$

for some $0 < \delta_p \le 1$ which is 1 for almost all p.

Using $\delta_\infty > 0$, we can show that there exists $0 < C < 1$ such that

$$\sum_{i=1}^n y_i^2 \ge C^2 (\sum_{i=1}^n y_i^2 + 1)$$

for any $(y_1, \cdots, y_n) \in \mathbf{V}(\mathbb{R})$. Hence

$$\begin{aligned} \mathrm{H}(x) &\ge C \prod_p |x_0|_p \sqrt{\sum_{i=1}^n \frac{x_i^2}{x_0^2} + 1} \cdot \prod_{p \in R_f} \max_{1 \le i \le n} \{ \frac{|x_i|_p}{|x_0|_p}, \delta_p \} \\ &\ge (C \prod_p \delta_p) \cdot \sqrt{\sum_{0 \le i \le n} x_i^2} \cdot \prod_{p \in R_f} \max_{0 \le i \le n} |x_i|_p \\ &= (C \prod_p \delta_p) \cdot \sqrt{\sum_{0 \le i \le n} x_i^2}. \end{aligned}$$

This proves the claim. □

Theorem 5.9. *Suppose that there are only finitely many* $\mathbf{G}(\mathbb{A})$*-orbits in* $\mathbf{V}(\mathbb{A})$. *Then for some* $a \in \mathbb{Q}_{>0}$ *and* $b \in \mathbb{Z}_{\ge 1}$,

$$\begin{aligned} &\#\{x \in \mathbf{V}(\mathbb{Q}) : \mathrm{H}(x) < T\} \\ &\asymp \#\{\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right) \in \mathbf{V}(\mathbb{Q}) : \sqrt{x_0^2 + \cdots + x_n^2} < T\} \\ &\asymp T^a (\log T)^{b-1}. \end{aligned}$$

Proof. To deduce this from Theorem 5.5, consider the embedding of GL_n into GL_{n+1} by $A \mapsto \mathrm{diag}(A, 1)$, and of $\mathbf{V}$ into the projective space $\mathbb{P}^n$ by $(x_1, \cdots, x_n) \mapsto [x_1 : \cdots : x_n : 1]$. This identifies $\mathbf{V}$ with the orbit $\mathbf{U} := [(v_0 : 1)]\mathbf{G}$ and $\mathrm{s} = x_{n+1}$ is an invariant section of the line bundle L obtained by pulling back $\mathcal{O}_{\mathbb{P}^n}(1)$, satisfying $\mathbf{U} = \{\mathrm{s} \ne 0\}$. Note for $x \in \mathbf{V}(\mathbb{Q})$,

$$\mathrm{H}_\mathrm{L}\left(\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0} : 1\right) = \mathrm{H}_\mathrm{L}(x_1 : \cdots : x_n : x_0) = \sqrt{x_0^2 + \cdots + x_n^2}.$$

Therefore this theorem is a special case of Corollary 5.5. □

When $\mathbf{G}$ is simply connected, we can replace $\asymp$ with $\sim$ in Theorem 5.4, and hence obtain the following example. Since $\mathbf{V} = \{x \in \mathrm{SL}_{2n} : x^t = -x\}$ is a homogeneous variety $\mathrm{Sp}_{2n} \backslash \mathrm{SL}_{2n}$ for the action $v.g = g^t v g$, we have:

Example 5.10. *Let* $n \ge 2$. *For some* $a \in \mathbb{Q}^+$, $b \in \mathbb{Z}_{\ge 0}$ *and* $c > 0$, *as* $T \to \infty$,

$$\#\{x \in \mathrm{SL}_{2n}(\mathbb{Q}) : x^t = -x, \max_{1 \le i,j \le 2n} \{|x_{ij}|, |x_0|\} < T\} \sim c \cdot T^a (\log T)^{b-1}.$$

where $x = \left(\frac{x_{ij}}{x_0}\right)$, $x_{ij} \in \mathbb{Z}$, $x_0 \in \mathbb{N}$ *and* $\mathrm{g.c.d}\{x_{ij}, x_0 : 1 \le i, j \le 2n\} = 1$.

S-integral points: We keep the same assumption on $\mathbf{V}$ from 5.2. Let S be a finite set of primes containing ∞, and consider the following S-height function on $\mathbf{V}_S := \prod_{p\in S}\mathbf{V}(\mathbb{Q}_p)$: for $x = (x_p)_{p\in S} \in \mathbf{V}_S$,

$$\mathrm{H}_S(x) := \prod_{p\in S} \|x_p\|_p, \tag{5.11}$$

where $\|\cdot\|_\infty$ is the Euclidean norm on $\mathbb{R}^n$, and $\|\cdot\|_p$ is the p-adic maximum norm on $\mathbb{Q}_p^n$.

Set

$$B_S(T) := \{x \in \mathbf{V}_S : \mathrm{H}_S(x) < T\}.$$

The following is obtained in [**3**, Prop. 8.11] for any $\mathbb{Q}$-algebraic group $\mathbf{G}$ and a closed orbit $\mathbf{V} = v_0\mathbf{G}$.

Theorem 5.12. *For any $v_0 \in \mathbf{V}_S$, the family $B_S(T) \cap v_0\mathbf{G}_S$ is well-rounded and*

$$\mathrm{vol}(B_S(T) \cap v_0\mathbf{G}_S) \asymp T^a(\log T)^b$$

for some $a \in \mathbb{Q}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$.

When $S = \{\infty\}$, we can replace $\asymp$ with $\sim$.

The notation $\mathbb{Z}_S$ is the subring of $\mathbb{Q}$ consisting of elements whose denominators are prime to all $p \notin S$. Hence if $S = \{\infty\}$, then $\mathbb{Z}_S = \mathbb{Z}$.

If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an S-congruence subgroup which preserves $\mathbf{V}(\mathbb{Z}_S)$, there are only finitely many Γ-orbits in $\mathbf{V}(\mathbb{Z}_S)$, say, $v_1\Gamma, \cdots, v_l\Gamma$. Set $\mathbf{L}_i = \mathrm{Stab}_{\mathbf{G}}(v_i)$ and $\mathbf{L}_{i,S} = \prod_{p\in S}\mathbf{L}_i(\mathbb{Q}_p)$.

Corollary 5.13. *Suppose that S is strongly isotropic for each $\mathbf{L}_i$. Then*

$$\#\{x \in \mathbf{V}(\mathbb{Z}_S) : \mathrm{H}_S(x) < T\} \asymp T^a(\log T)^b.$$

Proof. Since $\pi(\tilde{\mathbf{G}}_S)$ is a finite index normal subgroup of $\mathbf{G}_S$ for the simply connected cover $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$, its proof immediately reduces to the case when both $\mathbf{G}$ and $\mathbf{L}$ are simply connected groups.

Then Proposition 3.5 and Corollary 4.11 imply that

$$\#\{x \in \mathbf{V}(\mathbb{Z}_S) : \mathrm{H}_S(x) < T\} \sim_T \sum_{i=1}^{l} \frac{\mathrm{vol}((\Gamma \cap \mathbf{L}_{i,S})\backslash \mathbf{L}_{i,S})}{\mathrm{vol}(\Gamma\backslash\tilde{\mathbf{G}}_S)}\, \mathrm{vol}(B_S(T) \cap v_i\mathbf{G}_S). \tag{5.14}$$

Hence the claim follows from Theorem 5.12. $\square$

The asymptotic (5.14) in the case $S = \{\infty\}$ (and hence the integral points case) was proved by Eskin-Mozes-Shah [**27**] using the unipotent flows. Their result is more general since they also deal with maximal reductive (non-semisimple) $\mathbb{Q}$-groups. We believe Corollary 5.13 should hold only assuming that the stabilizer of v_0 is a reductive maximal $\mathbb{Q}$-subgroup, extending the argument [**27**] to the S-arithmetic setting. When $\mathbf{V}$ is symmetric and $S = \{\infty\}$, (5.14) was proved even earlier by Duke-Rudnick-Sarnak [**21**] and by Eskin-McMullen [**26**] (see section 8 for more discussion.)

5.3. Equidistribution. Suppose that $\mathbf{G}$ and $\mathbf{L}$ are simply connected. Then the connected components of $\mathbf{V}(\mathbb{R})$ are precisely $\mathbf{G}(\mathbb{R})$-orbits and $\mathbf{G}(\mathbb{A}_f)$ acts transitively on $\mathbf{V}(\mathbb{A}_f)$. Fix a compact subset $\Omega \subset v_0\mathbf{G}(\mathbb{R})$ with smooth boundary.

Corollary 5.15. *As $m \to \infty$, subject to $b_m \neq \emptyset$,*

$$\#\{x \in \mathbf{V}(\mathbb{Q}) : x \in \Omega,\ \textit{denominator of } x \textit{ is } m\} \sim \mathrm{vol}(\Omega) \times \mathrm{vol}(b_m)$$

where for $m = p_1^{k_1}\cdots p_r^{k_r}$

$$b_m = \{(x_p) \in \mathbf{V}(\mathbb{A}_f) : \|x_{p_i}\|_{p_i} = p_i^{k_i}\ \ \forall 1 \leq i \leq r,\ x_p \in \mathbf{V}(\mathbb{Z}_p) \quad \textit{for all } p \neq p_i\}.$$

Proof. The above follows from the observation that the family $\Omega \times b_m$ is W-well-rounded for any compact open subgroup W of $\mathbf{G}(\mathbb{A}_f)$ which preserves $\prod_{p\in R_f} \mathbf{V}(\mathbb{Z}_p)$ □

The above corollary implies that the rational points in $\mathbf{V}$ with denominator m are equidistributed on $\mathbf{V}(\mathbb{R})$ as $m \to \infty$.

6. Mixing and Hecke points

In this section, we will discuss an ergodic theoretic proof of the mixing of adelic groups as a special case of Theorem 4.1, and show that the adelic mixing is equivalent to the equidistribution of Hecke points together with the mixing of a finite product of corresponding local groups.

6.1. Mixing. We begin by recalling the notion of mixing in the homogeneous case. Let G be a locally compact second countable group and Γ a lattice in G. Let dx denote the probability invariant measure on $\Gamma\backslash G$. The group G acts on $L^2(\Gamma\backslash G)$ by $g\psi(x) = \psi(xg)$ for $g, x \in G, \psi \in L^2(\Gamma\backslash G)$.

Definition 6.1. The right translation action of G on the space $\Gamma\backslash G$ is called *mixing* if for any $\psi, \phi \in L^2(\Gamma\backslash G)$,

$$\langle g_i\psi, \phi\rangle = \int_{\Gamma\backslash G} \psi(xg_i)\phi(x)\, dx \to \int \psi dx \cdot \int \phi\, dx$$

for any $g_i \in G$ going to infinity.

We need the following well known lemma: we denote by $\Delta(G)$ the diagonal embedding of G into $G \times G$.

Lemma 6.2. *The following are equivalent:*

- *the right translation action of G on $\Gamma\backslash G$ is mixing;*
- *for any sequence $g \to \infty$ in G, the translate $\Delta(\Gamma)\backslash\Delta(G)(e, g)$ becomes equidistributed in $(\Gamma\times\Gamma)\backslash G \times G$.*

Proof. Observe that for $\psi, \phi \in C_c(\Gamma\backslash G)$,

$$\langle g\psi, \phi\rangle = \int_{x\in\Gamma\backslash G} \psi(xg)\phi(x)dx = \int_{(x,x)\in\Delta(\Gamma)\backslash\Delta(G)} (\psi\otimes\phi)(xg, x)dx.$$

Since the set of finite linear combinations of $\psi\otimes\phi$, $\psi, \phi \in C_c(\Gamma\backslash G)$ is dense in $C_c((\Gamma\times\Gamma)\backslash(G\times G))$, the claim follows. □

6.2. Hecke orbits. Denote by $\mathrm{Comm}(\Gamma) < G$ the commensurator group of Γ, that is, $a \in \mathrm{Comm}(\Gamma)$ if and only if $a\Gamma a^{-1}\cap\Gamma$ has a finite index both in Γ and $a\Gamma a^{-1}$.

Definition 6.3 (Hecke orbits). If $a \in \mathrm{Comm}(\Gamma)$,

$$\mathrm{T}_\Gamma(a) := \Gamma\backslash\Gamma a\Gamma$$

is called the Hecke orbit associated to a.

Using the bijection $\Gamma\backslash\Gamma a\Gamma = \Gamma\cap a^{-1}\Gamma a\backslash\Gamma$ given by $[a]\gamma \mapsto [\gamma]$, we have

$$\deg_\Gamma(a) := \#\,\mathrm{T}_\Gamma(a) = [\Gamma : \Gamma\cap a^{-1}\Gamma a].$$

Example 6.4. For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $a = \mathrm{diag}(p, p^{-1})$, $\Gamma\backslash\Gamma a\Gamma$ is in bijection with $\mathrm{SL}_2(\mathbb{Z}_p)a^{-1}\,\mathrm{SL}_2(\mathbb{Z}_p)/\,\mathrm{SL}_2(\mathbb{Z}_p) = \mathrm{SL}_2(\mathbb{Z}_p)a\,\mathrm{SL}_2(\mathbb{Z}_p)/\,\mathrm{SL}_2(\mathbb{Z}_p)$, that is, the $\mathrm{SL}_2(\mathbb{Z}_p)$-orbit of a^{-1} in the Bruhat-Tits tree. Hence $\mathrm{T}_\Gamma(a)$ corresponds to the $p(p+1)$ vertices in the $p+1$-regular tree of distance 2 from the vertex $x_0 := \mathbb{Z}_p\oplus\mathbb{Z}_p$. For $a = (\sqrt{p}, \sqrt{p}^{-1})$, $\mathrm{T}_\Gamma(a)$ gives $p+1$ vertices of distance 1 from x_0.

The following observation was first made in a paper by Burger and Sarnak [**9**].

LEMMA 6.5. *For a sequence* $a_i \in \mathrm{Comm}(\Gamma)$*, the following are equivalent:*

- *the Hecke orbit* $\mathrm{T}_\Gamma(a_i)$ *is equidistributed in* $\Gamma\backslash G$ *as* $i \to \infty$*, that is, for any* $\psi \in C_c(\Gamma\backslash G)$*,*
$$\frac{1}{\deg_\Gamma(a_i)} \sum_{x \in \mathrm{T}_\Gamma(a_i)} \psi(x) \to \int_{\Gamma\backslash G} \psi\, dx.$$
- *the orbit* $[(e, a_i^{-1})]\Delta(G)$ *becomes equidistributed in* $(\Gamma \times \Gamma)\backslash(G \times G)$ *as* $i \to \infty$*.*

PROOF. We use the homeomorphism between the space of Γ-invariant probability measures on $\Gamma\backslash G$ and the space of $\Delta(G)$-invariant probability measures on $(\Gamma \times \Gamma)\backslash(G \times G)$ given by $\mu \mapsto \tilde{\mu}$ where
$$\tilde{\mu}(f) = \int_{\Gamma\backslash G} \int_{\Gamma\backslash G} f(y, xy) d\mu(x) dy$$
(cf. [**4**, Prop. 8.1]). If μ_a is the probability measure which is the average of the dirac measures of the Hecke point $\mathrm{T}_\Gamma(a)$, then $\tilde{\mu}_a$ is the $\Delta(G)$-invariant measure supported on the orbit $(\Gamma \times \Gamma)\backslash(e, a_i^{-1})\Delta(G)$. Hence the claim follows. □

Moreover,
$$\deg_\Gamma(a) = \mathrm{vol}(\Gamma \cap a^{-1}\Gamma a \backslash G) = \mathrm{vol}((\Gamma \times \Gamma)\backslash(1, a)\Delta(G))$$
where the volumes are induced by the Haar measure on G which gives volume 1 for $\Gamma\backslash G$.

6.3. Adelic mixing. Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$-group. We will deduce the mixing of $\mathbf{G}(\mathbb{A})$ on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ from Theorem 4.1 for $\mathbf{G}$ simply connected and $\mathbb{Q}$-simple.

Fix a finite set S of primes, which contains ∞ if $\mathbf{G}(\mathbb{R})$ is non-compact.

Fixing an imbedding of $\mathbf{G}$ into GL_n, we set
$$\mathrm{H}_S(x) := \prod_{p \in S} \max |x(p)_{ij}|_p$$
where $x = (x(p))_{p \in S} \in \mathbf{G}_S$.

THEOREM 6.6. *Let* $\mathbf{G}$ *be connected simply connected and almost* $\mathbb{Q}$*-simple. The following equivalent statements hold for any* $\mathbf{G}$*-isotropic subset* S*:*

1. *The right translation action of* $\mathbf{G}(\mathbb{A})$ *on* $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ *is mixing.*
2. *For any* $g_i \to \infty$ *in* $\mathbf{G}(\mathbb{A})$*, the translate*
$$[(e, e)]\Delta(\mathbf{G}(\mathbb{A}))(e, g_i)$$
becomes equidistributed in $\tilde{X} \times \tilde{X}$*.*
3. *For any* S*-congruence subgroup* Γ *of* $\mathbf{G}(\mathbb{Q})$*,* $a_i \in \mathbf{G}(\mathbb{Q})$ *and* $x_i \in \mathbf{G}_S$ *such that* $\deg_\Gamma(a_i) \cdot \mathrm{H}_S(x_i) \to \infty$*, the closed orbit*
$$[(e, a_i)]\Delta(\mathbf{G}_S)(e, x_i)$$
becomes equidistributed on $(\Gamma \times \Gamma)\backslash(\mathbf{G}_S \times \mathbf{G}_S)$*.*
4. *For any* S*-congruence subgroup* Γ *of* $\mathbf{G}(\mathbb{Q})$*, and* $a_i \in \mathbf{G}(\mathbb{Q})$ *with* $\deg_\Gamma(a_i) \to \infty$*,*
   - *the Hecke orbit* $\mathrm{T}_\Gamma(a_i)$ *becomes equidistributed in* $\Gamma\backslash\mathbf{G}_S$*;*
   - *the right translation action of* $\mathbf{G}_S$ *on* $\Gamma\backslash\mathbf{G}_S$ *is mixing.*

PROOF. The condition $\mathbf{G}$ being $\mathbb{Q}$-simple implies that the diagonal embedding of $\mathbf{G}$ into $\mathbf{G} \times \mathbf{G}$ is a maximal connected $\mathbb{Q}$-group. Hence (2) follows from Theorem 4.1.

The equivalence between (2) and (3) comes from the basic lemma 4.9. (1) and (2) are equivalent by Lemma 6.2. (3) and (4) are equivalent by Lemmas 6.2 and 6.5. □

REMARK 6.7. Since (2) follows from Theorem 4.1 which is proved using the unipotent flows on S-arithmetic setting, we have obtained by the equivalence of (1) and (2) an ergodic theoretic proof of the adelic mixing.

For S as above, the mixing of $\mathbf{G}_S$ on $\Gamma\backslash\mathbf{G}_S$ is a well-known consequence of the Howe-Moore theorem [**39**] on the decay of matrix coefficients for $\mathbf{G}(\mathbb{Q}_p)$'s, $p \in S$.

Hence the above theorem says that the adelic mixing is a consequence of the Howe-Moore theorem for $\mathbf{G}_S$ together with the equidistribution of Hecke points for *all* S-congruence subgroups for some fixed isotropic subset S (and hence for all isotropic S).

The equidistribution of Hecke points was obtained with a rate in [**14**] except for one case of some $\mathbb{Q}$-anisotropic form of a special unitary group. This last obstruction was removed by Clozel soon afterwards [**13**]. For $S = \{\infty\}$, a different proof for the equidistribution was given in [**28**] (without rates), using a theorem of Mozes-Shah [**46**] on unipotent flows.

The adelic mixing theorem can also be deduced from the property of the automorphic spectrum $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ based on the work of [**14**], [**48**], [**13**] and this approach is explained in the paper [**33**] and gives a rate of convergence for the mixing.

7. Bounds toward the Ramanujan conjecture on the automorphic spectrum

In this section, we discuss a quantitative adelic mixing statement and how they can be understood in view of the Ramanujan conjecture concerning the automorphic spectrum. We refer to [**55**] and [**12**] for the background on the Ramanujan conjecture.

We assume that $\mathbf{G}$ is a connected and absolutely simple $\mathbb{Q}$-group (e.g., $\mathbf{G} = \mathrm{SL}_n$ or PGL_n).

Note that the right translation action of $\mathbf{G}(\mathbb{A})$ on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ defines a unitary representation ρ of $\mathbf{G}(\mathbb{A})$ on the Hilbert space $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$, and hence a unitary representation of $\mathbf{G}(\mathbb{Q}_p)$ for each $p \in R$. Roughly speaking, the automorphic dual $\hat{\mathrm{Aut}}(\mathbf{G})_p$ of $\mathbf{G}(\mathbb{Q}_p)$ is the closure in the unitary dual of $\mathbf{G}(\mathbb{Q}_p)$ of the subset consisting of all irreducible constituents of the unitary representation $\rho|_{\mathbf{G}(\mathbb{Q}_p)}$.

A most sophisticated form of the Ramanujan conjecture is an attempt to identify $\hat{\mathrm{Aut}}(\mathbf{G})_p$ in the unitary dual $\hat{\mathbf{G}}(\mathbb{Q}_p)$. In the case of $\mathbf{G} = \mathrm{PGL}_2$ over $\mathbb{Q}$, the Ramanujan conjecture says that for each p, any infinite dimensional irreducible representation in $\hat{\mathrm{Aut}}(\mathbf{G})_p$ is strongly $L^{2+\epsilon}(\mathrm{PGL}_2(\mathbb{Q}_p))$ for any $\epsilon > 0$.

DEFINITION 7.1.
- A unitary representation ρ of $\mathbf{G}(\mathbb{Q}_p)$ is strongly $L^{p+\epsilon}$ if there exists a dense subset of vectors v, w such that the matrix coefficient function $\mathbf{G}(\mathbb{Q}_p) \to \mathbb{C}$ defined by
$$g \mapsto \langle \rho(g)v, w\rangle$$
is $L^{p+\epsilon}(\mathbf{G}(\mathbb{Q}_p))$-integrable for all $\epsilon > 0$.
- A unitary representation ρ is tempered if ρ is strongly $L^{2+\epsilon}$.

By the classification of the unitary dual of $\mathrm{PGL}_2(\mathbb{Q}_p)$ we know that there exists an irreducible unitary representation ρ_p of $\mathrm{PGL}_2(\mathbb{Q}_p)$ which is not L^{m_p}-integrable for arbitrary large m_p. By forming a (restricted) tensor product $\otimes'_{p\in R}\rho_p$, one can construct an irreducible unitary representation of $\mathrm{PGL}_2(\mathbb{A})$. A point made by the Ramanujan conjecture is that such a unitary representation cannot arise as a non-trivial irreducible constituent of $L^2(\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A}))$ if any of m_p is strictly larger than 2.

The following theorem says that in the case of $\mathbb{Q}$-rank at least 2, there is an obstruction of forming such a unitary representation of $\mathbf{G}(\mathbb{A})$ already in the level of unitary dual of $\mathbf{G}(\mathbb{Q}_p)$.

THEOREM 7.2. *([**48**], [**47**]) Suppose that* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) \geq 2$ *(e.g.,* PGL_n*,* $n \geq 3$*). Then there exists a positive number* m *(explicit and independent of* p*) such that any infinite dimensional irreducible unitary representation of* $\mathbf{G}(\mathbb{Q}_p)$ *is strongly* L^m *for all* $p \in R$*.*

To each root α of a maximal $\mathbb{Q}_p$-split torus of G, we associate the algebraic subgroup H_α (isomorphic either to $\mathrm{PGL}_2(\mathbb{Q}_p)$ or to $\mathrm{SL}_2(\mathbb{Q}_p)$) generated by the one-dimensional root subgroups $N_{\pm\alpha}$. The key step of the proof is then to show that for any irreducible unitary representation ρ of $\mathbf{G}(\mathbb{Q}_p)$, the restriction to H_α is tempered. In showing this, the main tool is Mackey's theory on the unitary representations of the semi-direct product $\mathrm{PGL}_2(\mathbb{Q}_p) \ltimes U_p$ (or $\mathrm{SL}_2(\mathbb{Q}_p) \times U_p$) for some non-trivial unipotent algebraic group U_p. In the case when $\mathbb{Q}_p$-rank is at least 2, one can always find such U_p so that H_α sits inside $H_\alpha \ltimes U_p \subset \mathbf{G}(\mathbb{Q}_p)$. Once we have a bound for those H_α's, we make use of the properties of tempered representations to extend the bound to the whole group $\mathbf{G}(\mathbb{Q}_p)$ [**48**].

The following is obtained in [**33**]: Choose a height function on $\mathbf{G}(\mathbb{A})$, for instance,

$$\mathrm{H}(g) := \prod_{p \in R} \max |(g_p)_{ij}|_p \quad \text{for } g = (g_p)_p \in \mathbf{G}(\mathbb{A})$$

using some $\mathbb{Q}$-embedding of $\mathbf{G} \to \mathrm{SL}_n$.

We write

$$L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) = L^2_{00}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})) \oplus \hat{\bigoplus}_{\chi \in \Lambda} \mathbb{C}\chi$$

where Λ denotes the set of all automorphic characters. Hence the Hilbert space $L^2_{00}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ denotes the orthogonal complement to the subspace spanned by automorphic characters. If $\mathbf{G}$ is simply connected, Λ has only the trivial character.

THEOREM 7.3 (Adelic mixing). [**33**] *Fix a maximal compact subgroup* K *of* $\mathbf{G}(\mathbb{A})$*. There exists* $k > 0$ *(explicit) and* $c \geq 1$ *such that for any* K*-invariant* $\psi_1, \psi_2 \in L^2_{00}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$*, we have*

$$|\langle g\psi_1, \psi_2 \rangle| \leq c \cdot \mathrm{H}(g)^{-k} \|\psi_1\| \cdot \|\psi_2\| \quad \text{for all } g \in \mathbf{G}(\mathbb{A}).$$

We stated the above theorem only for K-invariant functions for simplicity. However the same holds for smooth functions as well, provided the L^2-norms of ψ_i are replaced by suitable Sobolev norms of ψ_i (see [**33**, Thm. 3.25] for details.) In fact, in most applications, we need the smooth version of Theorem 7.3.

EXAMPLE 7.4. Let $n \geq 3$. Let ψ_1, ψ_2 be $K := \mathrm{SO}_n \times \mathrm{SL}_n(\mathbb{Z}_f)$-invariant functions in $L^2(\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A}))$ with $\int \psi_i = 0$ and $\|\psi_i\| = 1$.

For any $\epsilon > 0$, there is $C = C_\epsilon > 0$ such that

$$|\langle g\psi_1, \psi_2 \rangle| \leq C \cdot \prod_p \prod_{i=1}^{[n/2]} \left(\frac{a_{p,i}}{a_{p,n+1-i}} \right)^{-1/2+\epsilon} \leq C \cdot \mathrm{H}(g)^{-1/2+\epsilon}$$

where $g = (g_p)$ and $g_p = \mathrm{diag}(a_{p,1}, \cdots, a_{p,n})$ with $a_{p,1} \geq \cdots \geq a_{p,n} > 0$ and $a_{\infty,i} \in \mathbb{R}^+$ and $a_{p,i} \in p^{\mathbb{Z}}$ for finite p.

The Hilbert space $L^2_{00}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ can be decomposed into the direct integral of irreducible unitary $\mathbf{G}(\mathbb{A})$-representations ρ and each ρ is of the form $\otimes'_{p \in R} \rho_p$ where ρ_p is an irreducible infinite dimensional unitary representation of $\mathbf{G}(\mathbb{Q}_p)$. Hence $\langle g\psi_1, \psi_2 \rangle$ is the direct integral of $\prod_p \langle \rho_p(g_p)\psi_{1p}, \psi_{2p} \rangle$'s. Now we are reduced to understanding the matrix coefficients $\langle \rho_p(g_p)\psi_{1p}, \psi_{2p} \rangle$ for an infinite dimensional irreducible unitary representation ρ_p of $\mathbf{G}(\mathbb{Q}_p)$.

Although we know the decay phenomenon for the matrix coefficients of unitary representations over each local field $\mathbb{Q}_p$ by the work of Howe-Moore [**39**], we need to have some uniformity of the decay over all p's, namely a weak form of the Ramanujan conjecture for $\mathbf{G}$. In the case of the $\mathbb{Q}$ rank is at least 2 and hence the $\mathbb{Q}_p$-rank of $\mathbf{G}$ is at least 2 for each p, this is achieved in [**48**]. In the case when the $\mathbb{Q}$-rank of $\mathbf{G}$ is one, the $\mathbb{Q}_p$-rank may be one or higher. When its $\mathbb{Q}_p$-rank is higher, one uses again [**48**] and when the $\mathbb{Q}_p$-rank is one, one now has to use more than the fact that ρ_p is an infinite dimensional unitary representation of $\mathbf{G}(\mathbb{Q}_p)$. Using the fact that ρ_p is indeed an automorphic representation, one uses the lifting of automorphic bound of $\mathrm{SL}_2(\mathbb{Q}_p)$ to $\mathbf{G}(\mathbb{Q}_p)$ due to Burger-Sarnak [**9**] and Clozel-Ullmo [**15**]. Finally when the $\mathbb{Q}$-rank and the $\mathbb{Q}_p$-rank of $\mathbf{G}$ are 0 and 1 respectively, Clozel analyzed what kind of automorphic representations occur in this situation and obtained a necessary bound for the decay [**13**], based on Jacquet-Langlands correspondence, and base changes of Rogawski and Clozel.

Remark 7.5. The bounds toward the Ramanujan conjecture we discuss in this section are very crude in many cases, not at all close to the optimal bounds (but they are optimal in the example 7.4 due to the continuous spectrum.) We point out that in recent applications to sieve, obtaining the Ramanujan bounds as close to optimal ones are very critical (see [**8**]).

We remark that using the volume computation for the adelic height balls made by Shalika, Takloo-Bighash and Tschinkel [**58**] one can deduce the following from 7.3:

Corollary 7.6. *The quasi-regular representation ρ of $\mathbf{G}(\mathbb{A})$ on $L^2_{00}(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ is strongly L^q for some explicit $q > 0$.*

We remark that the Ramanujan conjecture for PGL_2 implies that the representation $\rho := L^2_{00}(\mathrm{PGL}_2(\mathbb{Q})\backslash \mathrm{PGL}_2(\mathbb{A}))$ is strongly $L^{4+\epsilon}$: the conjecture implies that (cf. [**33**, proof of Thm 3.10]) for any $K := \mathrm{PO}_2 \times \prod_p \mathrm{PGL}_2(\mathbb{Z}_p)$-finite unit vectors v and w,

$$|\langle \rho(g)v, w\rangle| \le d_{v,w} \cdot \prod_{p\in R} \Xi_p(g_p) \quad \text{for } g = (g_p) \in \mathrm{PGL}_2(\mathbb{A})$$

where Ξ_p denotes the Harish-Chandra function of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and $d_{v,w}$ depends only on the dimensions of K-span of v and w.

From

$$\Xi_p \begin{pmatrix} p^k & 0 \\ 0 & 1 \end{pmatrix} = p^{-k/2}\left(\frac{k(p-1)+(p+1)}{p+1}\right),$$

we can deduce that for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that for $g = (g_p) \in \mathbf{G}(\mathbb{A})$,

$$\mathrm{H}^{-1/2}(g) \le \Xi(g) \le C_\epsilon\, \mathrm{H}(g)^{-1/2+\epsilon}$$

where $\Xi := \prod_p \Xi_p$, $\mathrm{H} = \prod_p \mathrm{H}_p$ and $\mathrm{H}_p(g_p)$ is the maximum p-adic norm of g_p.

Note that for any $\epsilon > 0$, there is $c_\epsilon > 0$ such that for any $\sigma > 0$,

$$\int_{\mathrm{PGL}_2(\mathbb{R})} \|g\|_\infty^{-\sigma/2} dg_\infty \le c_\epsilon \int_0^\infty e^{-t(\sigma/2)} e^{t(1+\epsilon)} dt.$$

Hence $\int_{\mathrm{PGL}_2(\mathbb{R})} \|g\|_\infty^{-s/2} dg_\infty$ absolutely converges for $\Re(s) > 2$.

Observe that

$$\begin{aligned}\int_{\mathrm{PGL}_2(\mathbb{Q}_p)} \mathrm{H}_p(g_p)^{-s/2} dg_p &= \sum_{k\geq 0} p^{-ks/2} \operatorname{vol}(\mathrm{PGL}_2(\mathbb{Z}_p) \begin{pmatrix} p^k & 0 \\ 0 & 1 \end{pmatrix} \mathrm{PGL}_2(\mathbb{Z}_p)) \\ &= 1 + \sum_{k\geq 1} p^{-ks/2} p^k (1+p^{-1}) \\ &= (1-p^{-(s/2-1)})^{-1}(1+p^{-s/2}) \\ &= \zeta_p\left(\frac{s}{2}-1\right)(1+p^{-s/2}).\end{aligned}$$

Therefore

$$\int_{\mathrm{PGL}_2(\mathbb{A})} \mathrm{H}(g)^{-s/2} dg = \int_{\mathrm{PGL}_2(\mathbb{R})} \|g\|_\infty^{-s/2} dg_\infty \times \prod_p (1+p^{-s/2}) \cdot \zeta(\frac{s}{2}-1).$$

Since the Riemann zeta function $\zeta(s)$ has a pole at $s = 1$ and $\prod_p(1+p^{-s/2})$ absolutely converges for $\Re(s) > 2$, the height zeta function $\mathcal{Z}(s) := \int_{\mathrm{PGL}_2(\mathbb{A})} \mathrm{H}(g)^{-s/2} dg$ has a meromorphic continuation to $\Re(s) > 2$ with an isolated pole at $s = 4$. In particular, for any $\epsilon > 0$, $\mathcal{Z}(4+\epsilon) < \infty$ and hence $\int_{\mathrm{PGL}_2(\mathbb{A})} \Xi(g)^{4+\epsilon}\, dg < \infty$, proving the claim.

Remark: By the equivalence of Lemma 6.2, we can deduce from the quantitative mixing theorem the equidistribution of the closed $\Delta(G)$-orbits $X_a := [(e, a^{-1})]\Delta(G)$ with respect to the Haar measure with the rate given by $\operatorname{vol}(X_a)^{-k}$ for some $k > 0$. Analogous statement is true even in the positive characteristic case since the results in [**48**] are valid.

A much more general result in this direction (characteristic zero case) was recently obtained in [**22**].

8. Counting via mixing and the wavefront property

Let G be a locally compact and second countable group, Γ a lattice in G and L a closed subgroup of G such that $L \cap \Gamma$ is a lattice in L.

Can the mixing of G on $\Gamma\backslash G$ be used to count points in the Γ-orbit $[e]\Gamma$ on $L\backslash G$? There are two alternative methods developed by Duke-Rudnick-Sarnak [**21**] and by Eskin-McMullen [**26**] which show that the answer is yes. Although both papers are based eventually on the spectral gap property (or the mixing property) of the group actions, the ways to use the spectral gap property are different.

In this section, we will present the methods in [**26**] which make use of a certain geometric property of $L\backslash G$, called the wavefront property.

Remark 8.1. We remark that the idea of using the mixing property in counting problems goes back to Margulis' 1970 thesis [**44**].

We give a slight variant of the wavefront property.

Definition 8.2. Let G be a locally compact group and L a closed subgroup of G.

- For a (non-compact) Borel subset E of G, the triple $(G, L : E)$ is said to have the wavefront property if for every neighborhood U of e in G, there exists a neighborhood V of e in G such that
$$LVg \subset LgU \quad \text{for all } g \in E.$$
- We say (G, L) has the wavefront property if $(G, L : E)$ has the wavefront property for some Borel subset $E \subset G$ satisfying $G = LE$.

It is easy to observe that if $(G, L : E)$ has the wavefront property, so does $(G, L : EK)$ for any compact subgroup K of G.

This property means roughly that the g-translate of a small neighborhood of the base point $z_0 := [L]$ in $L\backslash G$ remains near $z_0 g$ uniformly over all $g \in E$.

We give two examples for $G = \mathrm{SL}_2(\mathbb{R})$ below. For $a_t := \mathrm{diag}(e^{t/2}, e^{-t/2})$. let

$$A^+ = \{a_t : t \geq 0\} \quad \text{and} \quad A^- = \{a_t : t \leq 0\},$$

and $A = A^+ \cup A^-$. Let

$$N = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\} \quad \text{and} \quad N^- = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbb{R}\}.$$

Set $K = \mathrm{SO}_2 = \{x \in \mathrm{SL}_2(\mathbb{R}) : xx^t = e\}$.

Example 8.3. The triple $(\mathrm{SL}_2(\mathbb{R}), N : A^- K)$ has the wavefront property: Let U be an ϵ-neighborhood of e in $\mathrm{SL}_2(\mathbb{R})$. Since NAN^- is a Zariski dense open subset of G, we may assume that $U = (N \cap U)(A \cap U)(N^- \cap U)$. For any $a_t, a \in A$, and $n^- \in N^-$, observe that

$$(an^-)a_t = a_t a(a_t^{-1} n^- a_t).$$

Since the conjugation by a_t^{-1} is a contracting automorphism of N^- for any $t < 0$, we have $a_t^{-1}(N^- \cap U)a_t \subset N^- \cap U$. Therefore for $a_t \in A^-$,

$$NUa_t = N(A \cap U)(N^- \cap U)a_t \subset Na_t(A \cap U)(N^- \cap U) \subset Na_t U.$$

Hence the claim is proved.

Example 8.4. The pair $(\mathrm{SL}_2(\mathbb{R}), K)$ has the wavefront property: It suffices to prove that $(\mathrm{SL}_2(\mathbb{R}), K : A^+)$ has the wavefront property, since $G = KA^+K$ by the Cartan decomposition. Let U be an ϵ-neighborhood of e in $\mathrm{SL}_2(\mathbb{R})$. Using the Iwasawa decomposition $G = KAN$, we may assume that $U = (K \cap U)(A \cap U)(N \cap U)$. For any $a_t \in A^+$, since the conjugation by a_t is a contraction on N,

$$KUa_t = K(A \cap U)(N \cap U)a_t \subset Ka_t(A \cap U)a_t^{-1}(N \cap U)a_t \subset Ka_t U.$$

Hence the claim is proved.

Proposition 8.5. *Let $L < G$ be locally compact groups as above and $E \subset G$. Suppose the following:*

1. *The right translation action of G on $\Gamma\backslash G$ is mixing;*
2. *The wavefront property holds for $(G, L : E)$.*

Then, for any sequence $g_i \in E$ tending to ∞ in $L\backslash G$, the translate $\Gamma\backslash\Gamma L g_i$ becomes equidistributed in $\Gamma\backslash G$.

Proof. Let $Y = (\Gamma \cap L)\backslash L$ and $X = \Gamma\backslash G$. Denote by μ_L and μ_G the Haar measures on L and G which give one on Y and X respectively. For $\psi \in C_c(\Gamma\backslash G)$, we would like to show that

$$I_g := \int_Y \psi(yg)\, d\mu_L(y) \to \int_X \psi\, d\mu_G \quad \text{as } g \in E \text{ goes to infinity in } L\backslash G. \tag{8.6}$$

Suppose first that Y is compact. Then we can choose a Borel subset W in G transversal to L, so that the multiplication $m : Y \times W \to YW$ is a bijection onto its image $YW \subset X$. By the wavefront property, for any small neighborhood U of e in G, there exists W so that

YWg remains inside YgU for all $g \in E$. Hence by the uniform continuity of ψ, and by taking W small enough, we can assure that I_g is close to

$$\frac{1}{\mathrm{vol}(W)} \int_{YWg} \psi \, d\mu_G = \frac{1}{\mathrm{vol}(W)} \langle g\psi, \chi_{YW} \rangle$$

where χ_{YW} is the characteristic function of YW. It now follows from the mixing that

$$\frac{1}{\mathrm{vol}(W)} \langle g\psi, \chi_{YW} \rangle \sim \int_X \psi \, d\mu_G$$

as $g \to \infty$, and hence (8.6) holds. When Y is non-compact, such a W does not exist in general. In this case, we work with a big compact piece Y_ϵ of Y with co-volume less than ϵ. The above argument then gives that I_g is close to $\mu_L(Y_\epsilon) \int_X \psi d\mu_G$ for all large $g \in L\backslash E$. Since $\mu_L(Y \setminus Y_\epsilon) \leq \epsilon$ and $\|\psi\|_\infty$ is bounded, we can deduce (8.6). We refer to [**26**] for more details. In the case when Y is non-compact, the above modification is explained in [**3**]. □

Corollary 8.7. *The sequence* $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) a_t N$ *becomes equidistributed in the space* $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ *as* $t \to -\infty$.

Proof. Since a_t normalizes N,

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) a_t N = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}) N a_t.$$

Hence by the mixing of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ and Example 8.3, we can deduce the claim. □

This corollary can be interpreted as the equidistribution of long closed horocycles, since N-orbits are precisely horocycles in the identification of $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})$ with the unit tangent bundle of the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. This result was first obtained by Sarnak [**54**] with rates of convergence. The approach explained here has been well known and can be made effective using the quantitative mixing.

Putting Propositions 8.5 and 3.5 together, we state:

Corollary 8.8. *Let* $L < G$ *be locally compact groups as above and* $E \subset G$ *a Borel set. Let* $\{B_n \subset L\backslash G\}$ *be a sequence of compact subsets whose volume tending to infinity. Suppose the following:*

(1) *The right translation action of* G *on* $\Gamma\backslash G$ *is mixing.*
(2) *The wavefront property holds for* $(G, L : E)$.
(3) $\mathrm{vol}\,(B_n) \sim_n \mathrm{vol}(B_n \cap L\backslash LE)$ *and* $\{B_n\}$ *is well-rounded.*

Then for $x_0 = [L]$, *as* $n \to \infty$,

$$\# x_0\Gamma \cap B_n \sim \mathrm{vol}(B_n).$$

Using the well known fact that the Riemannian balls are well-rounded, we deduce the following from Example 8.4 and the above corollary:

Example 8.9. Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{R})$. Set $x_0 = i \in \mathbb{H} = \{x + iy : y > 0\}$ and let d denote the hyperbolic distance. Let B_t denote the ball of radius t centered at x_0. We then have

$$\{x \in x_0\Gamma : d(x, x_0) < t\} \sim \mathrm{vol}(B_t).$$

8.1. Affine symmetric variety. We now discuss more general examples of G and L for which Corollary 8.8 can be applied. These are affine symmetric pairs, which are generalizations of Riemannian symmetric pairs.

Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$-group. A $\mathbb{Q}$-subgroup $\mathbf{L}$ is called symmetric if there is an involution σ of $\mathbf{G}$ defined over $\mathbb{Q}$ such that $\mathbf{L} = \{g \in \mathbf{G} : \sigma(g) = g\}$.

THEOREM 8.10. *Let $\mathbf{L}$ be a symmetric subgroup of $\mathbf{G}$. For any finite set of primes S, the pair $(\mathbf{G}_S, \mathbf{L}_S)$ satisfies the wavefront property.*

PROOF. The proof easily reduces to the case when S is a singleton. When $S = \{\infty\}$, this was proved by Eskin-McMullen [**26**]. Their proof was based on the Cartan decomposition for real symmetric spaces. The claim is obtained in [**3**] for $S = \{p\}$ based on the Cartan decomposition for p-adic symmetric spaces ([**5**] and [**19**]). □

Let $\mathbf{V}$ be an affine symmetric variety defined over $\mathbb{Q}$, i.e., $\mathbf{V} = v_0\mathbf{G}$ where $\mathbf{V} \subset \mathrm{SL}_n$ is a $\mathbb{Q}$-embedding and $v_0 \in \mathbb{Q}^n$ and the stabilizer $\mathbf{L}$ of v_0 is a symmetric subgroup of $\mathbf{G}$. Let S be a finite set of primes which contains ∞ if $\mathbf{G}(\mathbb{R})$ is non-compact, and consider the height function H_S on $\mathbf{V}_S = \prod_p \mathbf{V}(\mathbb{Q}_p)$ as in 5.11. Set

$$B_S(T) := \{x \in \mathbf{U}_S : \mathrm{H}_S(x) < T\}.$$

THEOREM 8.11. *Assume that $\mathbf{G}$ is $\mathbb{Q}$-simple. As $T \to \infty$,*

$$\#\{x \in \mathbf{V}(\mathbb{Z}_S) : \mathrm{H}_S(x) < T\} \sim \mathrm{vol}(B_S(T)) \asymp T^a (\log T)^b$$

for some $a \in \mathbb{Q}_{>0}$ and $b \in \mathbb{Z}_{\geq 0}$.

PROOF. Let Γ be an S-congruence subgroup preserving $\mathbf{V}(\mathbb{Z}_S)$. It suffices to obtain the asymptotic for $\# v_0\Gamma \cap B_S(T)$ assuming that $\mathbf{G}$ is simply connected (see the proof of Corollary 5.13). So we have the mixing of the right translation action of $\mathbf{G}_S$ on $\Gamma\backslash\mathbf{G}_S$. By Theorem 5.12, for each $v_0 \in \mathbf{V}_S$, the family $v_0\mathbf{G}_S \cap B_S(T)$ is well-rounded. Hence the claim follows from Theorems 8.8 and 8.10. □

We note that in this theorem there is no restriction on S and the stabilizer may not be semisimple, unlike Corollary 5.13. As mentioned in section 5, Theorem 8.11 in the case $S = \{\infty\}$ was proved in [**21**] and [**26**] without an explicit computation of the asymptotic growth of $\mathrm{vol}(B_\infty(T))$ in general. The asymptotic $\mathrm{vol}(B_\infty(T)) \sim cT^a(\log T)^b$ was obtained independently in [**45**] and [**36**] for group varieties, and [**37**] for general symmetric varieties.

The main advantage of using the mixing property in counting problems is its effectiveness. In fact, the above theorem is shown effectively in [**3**], by obtaining the effective versions of (2) and (3) in Theorem 8.8. We note that this was done in [**45**] in the case of group varieties (see also [**34**]).

Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an S-congruence subgroup. Set $X_S := \Gamma\backslash\mathbf{G}_S$ and $Y_S := \Gamma \cap \mathbf{L}_S\backslash\mathbf{L}_S$. Also set $S_f := S \setminus \{\infty\}$.

DEFINITION 8.12. We say that the translate $Y_S g$ becomes effectively equidistributed in X_S as $g \to \infty$ in $\mathbf{L}_S\backslash\mathbf{G}_S$ if there exist $m \in \mathbb{N}$ and $r > 0$ such that, for any compact open subgroup W of $\mathbf{G}_{S_f}$ and any compact subset C of X_S, there exists $c = c(W, C) > 0$ satisfying that for any $\psi \in C_c^\infty(X_S)^W$ with support in C, one has for all $g \in \mathbf{G}_S$

$$\left|\int_{Y_S} \psi(yg)\, d\mu_{Y_S}(y) - \int_{X_S} \psi\, d\mu_{X_S}\right| \leq c \cdot \mathcal{S}_m(\psi)\ \mathrm{H}_S(v_0 g)^{-r}$$

where $\mathcal{S}_m(\psi)$ depends only on the L^2-Sobolev norm of ψ of order m at ∞.

DEFINITION 8.13. A sequence of Borel subsets B_n in $\mathbf{V}_S$ is said to be effectively well-rounded if

(1) it is invariant under a compact open subgroup W of $\mathbf{G}_{S_f}$,
(2) there exists $\kappa > 0$ such that, uniformly for all $n \geq 1$ and all $0 < \epsilon < 1$,
$$\mathrm{vol}(B^+_{n,\epsilon} - B^-_{n,\epsilon}) = O(\epsilon^\kappa \,\mathrm{vol}(B_n))$$
where $B^+_{n,\epsilon} = B_n U_\epsilon W$ and $B^-_{n,\epsilon} = \cap_{u \in U_\epsilon W} B_n u$, and U_ϵ denotes the ball of center e and radius ϵ in $\mathbf{G}(\mathbb{R})$.
(3) for any $k > 0$, there exists $\delta > 0$ such that, uniformly for all large $n > 1$ and all $0 < \epsilon < 1$, one has
$$\int_{B^+_{n,\epsilon}} \mathrm{H}_S^{-k}(z)\, dz = O(\mathrm{vol}(B_n)^{1-\delta}).$$

It is not hard to adapt the proof of Proposition 3.5 to prove the following:

PROPOSITION 8.14. *Suppose that*

(1) *the translate $Y_S g$ becomes effectively equidistributed in X_S as $g \to \infty$ in $\mathbf{L}_S \backslash \mathbf{G}_S$;*
(2) *a sequence $\{B_n \subset \mathbf{V}_S\}$ of Borel subsets is effectively well-rounded and $\mathrm{vol}(B_n) \to \infty$.*

Then there exists a constant $\delta > 0$ such that
$$\# v_0 \Gamma \cap B_n = \mathrm{vol}(B_n)(1 + O(\mathrm{vol}(B_n)^{-\delta})).$$

THEOREM 8.15. [**3**] *Let S be a finite set of primes including ∞. In the same setup as in Theorem 8.11, there exists $\delta > 0$ such that*
$$\#\{x \in \mathbf{V}(\mathbb{Z}_S) : \mathrm{H}_S(x) < T\} \sim_T \mathrm{vol}(B_S(T))(1 + \mathrm{vol}(B_S(T))^{-\delta}).$$

Besides the effective equidistribution of the translates $\Gamma \backslash \Gamma \mathbf{L}_S g$ for symmetric pairs $(\mathbf{G}, \mathbf{L})$, we need the effective well-roundedness of the height balls, which was obtained in the complete generality of a homogeneous variety in [**3**, Prop. 14.2]. This subtle property of the height balls was properly addressed first in [**45**] for group varieties.

9. A problem of Linnik: Representations of integers by an invariant polynomial II

Let f be an integral homogeneous polynomial of degree d in n-variables, and consider the level sets
$$\mathbf{V}_m := \{x \in \mathbb{C}^n : f(x) = m\} \quad \text{for } m \in \mathbb{N}.$$
Then $\mathbf{V}_m(\mathbb{Z}) := \mathbf{V}_m \cap \mathbb{Z}^n$ is precisely the set of integral vectors representing m by f. Linnik asked a question on whether the radial projection of $\mathbf{V}_m(\mathbb{Z})$ on $\mathbf{V}_1(\mathbb{R})$ becomes equidistributed as $m \to \infty$ [**40**]. In the case when $\mathbf{V}_m$ is a homogeneous space of a semisimple algebraic group, this question has been studied intensively in recent years (see for instance, [**56**], [**20**], [**14**], [**32**] [**28**], [**29**], [**49**], [**24**], [**23**], [**22**], etc.,)

In this section, we discuss a generalization of the main results of Eskin-Oh in [**29**], which explains the title of this section. To formulate our results, denote by $\mathrm{pr}_\infty : \mathbf{V}_m(\mathbb{R}) \to \mathbf{V}_1(\mathbb{R})$ the radial projection given by $\mathrm{pr}_\infty(x) = m^{-1/d}x$. For a subset Ω of $\mathbf{V}_1(\mathbb{R})$, set

$$N_m(f, \Omega) := \# \,\mathrm{pr}_\infty(\mathbf{V}_m(\mathbb{Z})) \cap \Omega. \tag{9.1}$$

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group with a given $\mathbb{Q}$-embedding $\mathbf{G} \subset \mathrm{GL}_n$ and a non-zero vector $v_0 \in \mathbb{Q}^n$ such that
$$v_0 \mathbf{G} = V_1.$$

We assume that $\mathbf{L} := \mathrm{Stab}_{\mathbf{G}}(v_0)$ is a semisimple maximal connected $\mathbb{Q}$-group.

Connected components of $\mathbf{V}_1(\mathbb{R})$ are precisely the orbits of the identity component $\mathbf{G}(\mathbb{R})^\circ$. On each connected component $\mathcal{O}$, fix a $\mathbf{G}(\mathbb{R})^\circ$-invariant measure with respect to which the volumes of subsets of $\mathcal{O}$ are computed below.

THEOREM 9.2. *Fix a connected component $\mathcal{O}$ of $\mathbf{V}_1(\mathbb{R})$. As $m \to \infty$ along primes, the projection $\mathrm{pr}_\infty(\mathbf{V}_m(\mathbb{Z}))$ becomes equidistributed on $\mathcal{O}$, provided $N_m(f, \mathcal{O}) \neq 0$.*

The equidistribution in the above means that for any compact subsets $\Omega_1, \Omega_2 \subset \mathcal{O}$ of boundary measure zero and of non-empty interior, we have

$$\frac{N_m(f, \Omega_1)}{N_m(f, \Omega_2)} \sim \frac{\mathrm{vol}(\Omega_1)}{\mathrm{vol}(\Omega_2)}.$$

In the case when $\mathbf{L}$ has no compact factors over the reals, this theorem was obtained in [**29**] for any $m \to \infty$ provided $\mathrm{pr}_\infty(\mathbf{V}_m(\mathbb{Z}))$ has no constant infinite subsequence, which is clearly a necessary condition.

REMARK 9.3. Since we allow $m \to \infty$ only along the primes, the above theorem is weaker than what is desired. We think that our argument can be modified to obtain the equidistribution as long as the sequence m is co-prime to a fixed prime number, by proving a suitable generalization of [**17**, Thm. 3] in the S-arithmetic setting.

EXAMPLE 9.4. Fix $n \geq 3$. Let

$$\mathbf{V}_m := \{x \in \mathrm{M}_n : x = x^t, \quad \det(x) = m\}.$$

Then the projection of $\mathbf{V}_m(\mathbb{Z})$ to $\mathbf{V}_1(\mathbb{R})$ becomes equidistributed as $m \to \infty$ along primes.

In this example, $\mathbf{V}_1$ is a finite union of homogeneous spaces $\mathrm{SO}(p,q)\backslash \mathrm{SL}_n$ where $0 \leq p, q \leq n$ ranges over non-negative integers such that $p \leq q$ and $p+q = n$. Since $\mathrm{SO}(p,q)$'s are maximal connected subgroups of SL_n, and the sequence $\{\mathrm{pr}_\infty(\mathbf{V}_m(\mathbb{Z})) \cap \mathcal{O}\}$ is non-empty for each connected component $\mathcal{O}$ of $\mathbf{V}_1(\mathbb{R})$, the claim follows from Theorem 9.2.

We now discuss the proof of Theorem 9.2. The proof makes use the p-adic unipotent flows. Using the idea of dynamics in the homogeneous space of p-adic groups by extending the homogeneous space of $\mathbf{G}(\mathbb{R})$ to that of $\mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{Q}_p)$ was already implicit in the work of Linnik [**41**], as pointed out in [**23**]. This idea was also used in [**24**] and [**23**].

Since $\mathbf{G}(\mathbb{R})^\circ$ is equal to $\pi(\tilde{\mathbf{G}}(\mathbb{R}))$ where $\pi : \tilde{\mathbf{G}} \to \mathbf{G}$ is the simply connected cover, we may assume without loss of generality that $\mathbf{G}$ is simply connected.

Choose p which is strongly isotropic for $\mathbf{L}$. We denote by $\mathbb{Z}_p^*$ the group of p-adic units. Then $\mathbb{Z}_p^*$ is the disjoint union $\cup_{i=1}^k u_i(\mathbb{Z}_p^*)^d$ where d is the degree of f and $\{x \in \mathbb{Z}_p^* : x^d = 1\} = \{u_1, \cdots, u_k\}$. For $m \in \mathbb{N}$ with $(m,p) = 1$, choose $\alpha_m \in \mathbb{Z}_p^*$ such that $u_i/m = \alpha_m^d$ for some $1 \leq i \leq k$. We then define a projection

$$\mathrm{pr} : \cup_{(m,p)=1} \mathbf{V}_m(\mathbb{Q}) \to \mathbf{V}_1(\mathbb{R}) \times (\cup_i \mathbf{V}_{u_i}(\mathbb{Q}_p))$$

by

$$\mathrm{pr}(x) = (\mathrm{pr}_\infty(x), \mathrm{pr}_p(x)) = (m^{-1/d}x, \alpha_m x).$$

Set $I_i := \{m \in \mathbb{N} : p \nmid m, m \in u_i(\mathbb{Z}_p^*)^d\}$ so that $m \in I_i$ means $\mathrm{pr}(\mathbf{V}_m(\mathbb{Q})) \subset \mathbf{V}_1(\mathbb{R}) \times \mathbf{V}_{u_i}(\mathbb{Q}_p)$.

Note that for $\Omega \subset \mathbf{V}_1(\mathbb{R})$ and for $m \in I_i$,

$$\# \mathrm{pr}_\infty(\mathbf{V}_m(\mathbb{Z})) \cap \Omega = \# \mathrm{pr}(\mathbf{V}_m(\mathbb{Z}[p^{-1}]) \cap (\Omega \times \mathbf{V}_{u_i}(\mathbb{Z}_p)).$$

Therefore Theorem 9.2 follows from the equidistribution of $\mathrm{pr}(\mathbf{V}_m(\mathbb{Z}[p^{-1}])$ on each $\mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{Q}_p)$-orbit in $\cup_i(\mathbf{V}_1(\mathbb{R}) \times \mathbf{V}_{u_i}(\mathbb{Q}_p))$.

Set $S = \{\infty, p\}$, and let $\Gamma \subset \mathbf{G}(\mathbb{Z}[p^{-1}])$ be an S-congruence subgroup preserving $\mathbf{V}(\mathbb{Z}[p^{-1}])$. Let μ_G and μ_L be the invariant probability measures on $\Gamma\backslash\mathbf{G}_S$ and $\Gamma \cap \mathbf{L}_S\backslash\mathbf{L}_S$ respectively, and for each $\mathbf{G}_S$ orbit $\mathcal{O}_S$, let $\mu_{\mathcal{O}_S}$ denote the invariant measure on $\mathcal{O}_S$ compatible with μ_G and μ_L.

Fix $1 \le i \le k$ and for a $\mathbf{G}_S$-orbit $\mathcal{O}_S = \mathcal{O}_\infty \times \mathcal{O}_p$ in $\mathbf{V}_1(\mathbb{R}) \times \mathbf{V}_{u_i}(\mathbb{Q}_p)$ which contains $\mathrm{pr}(\xi_0)$ for some $\xi_0 \in \mathbf{V}(\mathbb{Q})$, we define for each $\xi \in \mathbf{V}_m(\mathbb{Q})$ with $\mathrm{pr}(\xi) \in \mathcal{O}_S$,

$$\omega(\xi) := \frac{\mu_L(g_\xi^{-1}\Gamma g_\xi \cap \mathbf{L}_S\backslash\mathbf{L}_S)}{\mu_G(\Gamma\backslash\mathbf{G}_S)}$$

where $g_\xi \in \mathbf{G}_S$ such that $\mathrm{pr}(\xi_0)g_\xi = \mathrm{pr}(\xi)$.

Proposition 9.5. *For any sequence* $\mathrm{pr}(\xi_m) \in \mathrm{pr}(\mathbf{V}_m(\mathbb{Z}[p^{-1}])) \cap \mathcal{O}_S$ *and as* $m \to \infty$ *along* $(m, p) = 1$*, the sequence* $\mathrm{pr}(\xi_m)\Gamma$ *is equidistributed on* $\mathcal{O}_S$ *unless it contains a constant infinite subsequence.*

In fact, for any $\psi \in C_c(\mathcal{O}_S)$*,*

$$\sum_{x \in \mathrm{pr}(\xi_m)\Gamma} \psi(x) \sim \omega(\xi_m) \cdot \int \psi \, d\mu_{\mathcal{O}_S}. \tag{9.6}$$

Proof. For simplicity, let $g_m = g_{\xi_m}$. By the duality [**29**] it suffices to prove that the closed orbit $\Gamma\backslash\Gamma g_m \mathbf{L}_S$ becomes equidistributed in $\Gamma\backslash\mathbf{G}_S$. Set $\mathbf{L}_m$ to be the stabilizer of ξ_m in $\mathbf{G}$. Then

$$\mathbf{L}_m(\mathbb{R}) \times \mathbf{L}_m(\mathbb{Q}_p) = g_m(\mathbf{L}(\mathbb{R}) \times \mathbf{L}(\mathbb{Q}_p))g_m^{-1}.$$

In particular, $\mathbf{L}_m(\mathbb{Q}_p)$ is conjugate to $\mathbf{L}(\mathbb{Q}_p)$ by an element of $\mathbf{G}(\mathbb{Q}_p)$. Therefore p is strongly isotropic for all $\mathbf{L}_m$. It follows from Theorem 4.10 that if the equidistribution (9.6) we desire does not hold, then by passing to a subsequence, there exist m_0 and $\{\delta_m \in \Gamma : m \ge m_0\}$ such that

$$\mathbf{L}_m = \delta_m^{-1}\mathbf{L}_{m_0}\delta_m.$$

Since $\mathbf{L}$ has finite index in its normalizer, this means, by passing to a subsequence, the existence of $\gamma'_m \in \Gamma$ such that

$$g_l^{-1}\gamma'_m g_m \in \mathbf{L}(\mathbb{R}) \times \mathbf{L}(\mathbb{Q}_p) \quad \text{for all large } m, l$$

and hence $\mathrm{pr}(\xi_m)\Gamma = \mathrm{pr}(\xi_l)(\Gamma)$ for all large m and l in I_i. This is contradiction. □

For each connected component $\mathcal{O}_\infty$ of $\mathbf{V}_1(\mathbb{R})$, and $m \in I_i$, we define

$$\omega_m(\mathcal{O}_\infty) := \sum_{\xi_m \in \Omega_m} \omega(\xi_m) \cdot \mu_{\mathcal{O}_p}(\mathbf{V}_{u_i}(\mathbb{Z}_p) \cap \mathcal{O}_p)$$

where Ω_m is the set of representatives of Γ-orbits in $\mathbf{V}_m(\mathbb{Z}[p^{-1}])$, $\mathcal{O}_p = \mathrm{pr}_p(\xi_m)\mathbf{G}(\mathbb{Q}_p)$ and the measure $\mu_{\mathcal{O}_p}$ is determined so that its product with $\mu_{\mathcal{O}_\infty}$ is compatible with μ_G and μ_L.

Theorem 9.2 follows from:

Theorem 9.7. *For any connected component* $\mathcal{O}_\infty$ *of* $\mathbf{V}_1(\mathbb{R})$*, and for any compact subset* $\Omega \subset \mathcal{O}_\infty$ *of boundary measure zero, we have*

$$N_m(f, \Omega) \sim \omega_m(\mathcal{O}_\infty) \cdot \mu_{\mathcal{O}_\infty}(\Omega)$$

if $m \to \infty$ *along primes, and* $N_m(f, \mathcal{O}_\infty) \neq 0$.

Proof. First note that $\mathrm{pr}(\mathbf{V}_m(\mathbb{Q})) \cap \mathrm{pr}(\mathbf{V}_l(\mathbb{Q})) = \emptyset$ for any primes m and l.

Therefore Proposition 9.5 implies that for any compact subset $\Omega \subset \mathcal{O}_\infty$ of smooth boundary, as $m \to \infty$ in I_i along primes,

$$\begin{aligned}\#\,\mathrm{pr}(\xi_m)\Gamma \cap (\Omega \times \mathbf{V}_{u_i}(\mathbb{Z}_p)) &\sim \omega(\xi_m)\mu_{\mathcal{O}_S}(\Omega \times (\mathbf{V}_{u_i}(\mathbb{Z}_p) \cap \mathcal{O}_p)) \\ &= \omega(\xi_m) \cdot \mu_{\mathcal{O}_\infty}(\Omega) \cdot \mu_{\mathcal{O}_p}(\mathbf{V}_{u_i}(\mathbb{Z}_p) \cap \mathcal{O}_p).\end{aligned}$$

Since for $m \in I_i$

$$\#\,\mathrm{pr}(\mathbf{V}_m(\mathbb{Z})) \cap \Omega = \sum_{\xi_m \in \Omega_m} \#\,(\mathrm{pr}(\xi_m)\Gamma \cap (\Omega \times \mathbf{V}_{u_i}(\mathbb{Z}_p)))$$

the claim follows.

□

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