Orbit closures of the $SL_2(\mathbb{R})$-action and Kleinian manifolds

Hee Oh

Yale University

Joint work with McMullen and Mohammadi

Comments/questions from Maryam

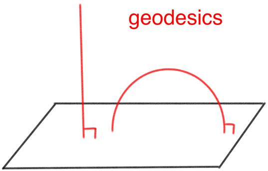
- (Princeton, 2006) I didn't expect to understand any, but I understood most of it.
- (Princeton, 2009) I thought Roblin studied these questions.
- (Stanford, 2015) Relation between isoperimetric constants and resonance-free planes for convex cocompact surfaces?
- (MSRI, 2015) Can you say something about a sequence of H-orbits?

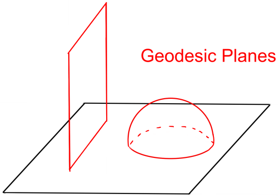
We will begin by discussing:

geodesic planes in hyperbolic 3-manifolds

Upper half space model of Hyperbolic 3-space

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\}, \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}$$





- $PSL_2(\mathbb{C})$ acts on $\partial(\mathbb{H}^3) = \hat{\mathbb{C}}$ by Möbius transformations;
- $PSL_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ (Poincare ext. thm)

Definition

A Kleinian group is a (torsion-free) discrete subgp of $\mathrm{PSL}_2(\mathbb{C})$.

A complete hyperbolic 3-mfld M can be presented as

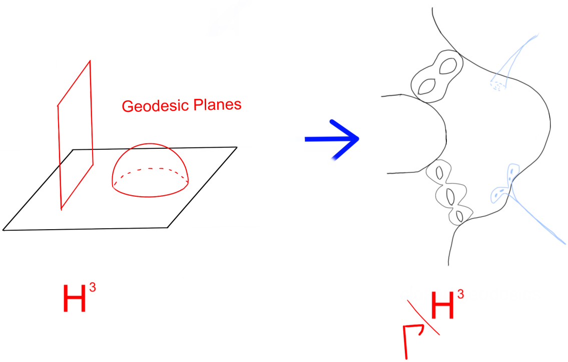
$$M = \Gamma \backslash \mathbb{H}^3.$$

for a Kleinian group Γ.

Geodesic planes in $M = \Gamma\backslash\mathbb{H}^3$

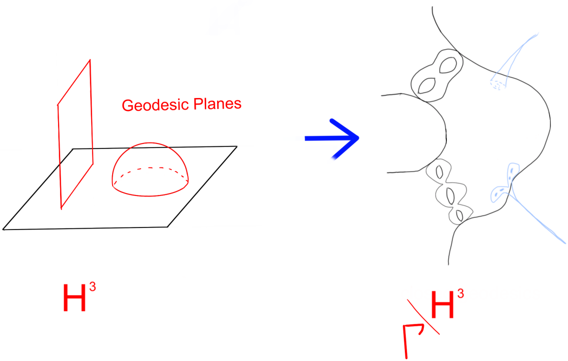
Definition

A geodesic plane in M is a totally geodesic immersion of $\mathbb{H}^2$ in M.



Question

- Can we classify all possible closures of geod. planes in *M*?
- Are all possible closures are submanifolds of *M*?



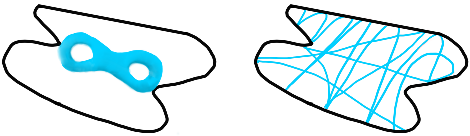
Closed-Dense dichotomy

Theorem (Ratner, Shah 1991)

Let Vol$(M) < \infty$*. Any geodesic plane* $P \subset M$ *is*

- *closed; or*
- *dense*

Moreover, a closed P *is properly immersed and has finite area.*



This theorem applies only to countably many hyp manifolds by Mostow rigidity theorem.

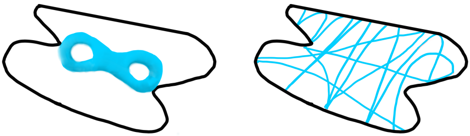
Closed-Dense dichotomy

Theorem (Ratner, Shah 1991)

Let $\mathrm{Vol}(M) < \infty$. *Any geodesic plane* $P \subset M$ *is*

- *closed; or*
- *dense*

Moreover, a closed P *is properly immersed and has finite area.*

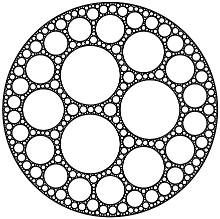


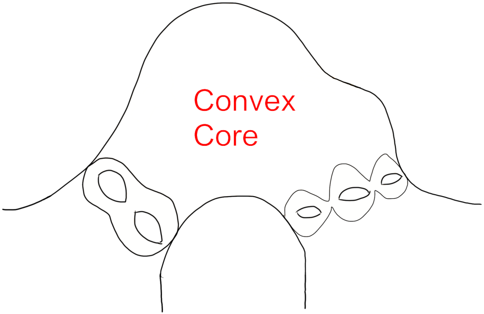
This theorem applies only to countably many hyp manifolds by Mostow rigidity theorem.

Question

To what extent does this kind of the rigidity theorem persist in the infinite volume hyperbolic manifolds?

The limit set Λ and the convex core of $M = \Gamma\backslash\mathbb{H}^3$ play important roles.

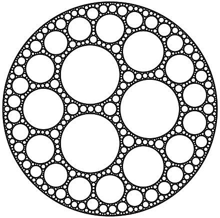




Definition (Limit set of Γ)

$\Lambda(\Gamma)$: the set of all accum. pts of $\Gamma(z)$ for any $z \in \hat{\mathbb{C}}$

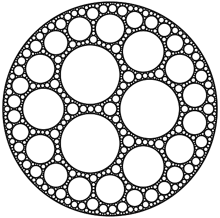
If $\mathrm{Vol}(M) < \infty$, $\Lambda = \hat{\mathbb{C}}$.
In general, Λ is a fractal set.



Definition (Limit set of Γ)

$\Lambda(\Gamma)$: the set of all accum. pts of $\Gamma(z)$ for any $z \in \hat{\mathbb{C}}$

If $\mathrm{Vol}(M) < \infty$, $\Lambda = \hat{\mathbb{C}}$.
In general, Λ is a fractal set.

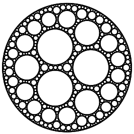


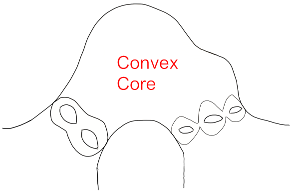
Definition (Convex core of $M = \Gamma\backslash\mathbb{H}^3$)

$$\mathrm{core}(M) := \Gamma\backslash\, \mathrm{hull}(\Lambda) \subset \Gamma\backslash\mathbb{H}^3 = M;$$

where hull(Λ) is the smallest convex subset of $\mathbb{H}^3$ cont. all geodesics connecting pts in Λ.

If $\mathrm{Vol}(M) < \infty$, $M = \mathrm{core}(M)$.



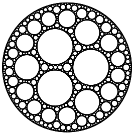


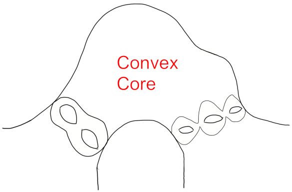
Definition (Convex core of $M = \Gamma \backslash \mathbb{H}^3$)

$$\mathrm{core}(M) := \Gamma \backslash \, \mathrm{hull}(\Lambda) \subset \Gamma \backslash \mathbb{H}^3 = M;$$

where hull(Λ) is the smallest convex subset of $\mathbb{H}^3$ cont. all geodesics connecting pts in Λ.

If $\mathrm{Vol}(M) < \infty$, $M = \mathrm{core}(M)$.



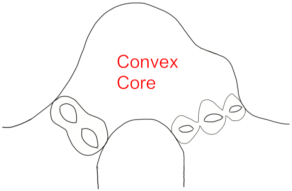


Definition

- M is geometrically finite iff Vol(1-nbd of core(M)) $< \infty$;
- M is convex cocompact if core(M) is compact.

In the following, we assume

M is convex cocompact, non-fuchsian and $\mathrm{Vol}(M) = \infty$.

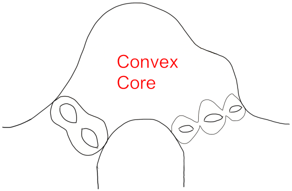


Definition

- M is geometrically finite iff Vol(1-nbd of core(M)) $< \infty$;
- M is convex cocompact if core(M) is compact.

In the following, we assume

M is convex cocompact, non-fuchsian and Vol(M) $= \infty$.



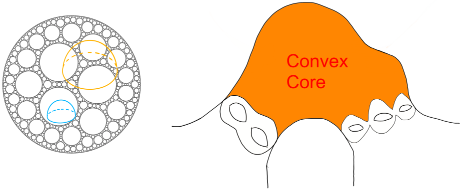
Set

$$M^* := \text{the interior of core}(M) \neq \emptyset$$

so $M - M^* =$ end components of M.

There are two kinds of planes:

- P with $P \cap M^* \neq \emptyset$;
- P with $P \cap M^* = \emptyset$.



If $P \cap M^* = \emptyset$, $\overline{P} \subset M - M^*$.

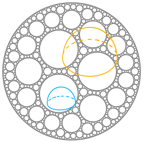
Set

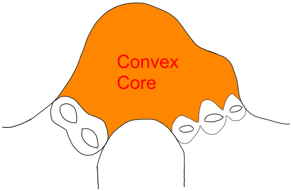
$$M^* := \text{the interior of core}(M) \neq \emptyset$$

so $M - M^* =$ end components of M.

There are two kinds of planes:

- P with $P \cap M^* \neq \emptyset$;
- P with $P \cap M^* = \emptyset$.





If $P \cap M^* = \emptyset$, $\overline{P} \subset M - M^*$.

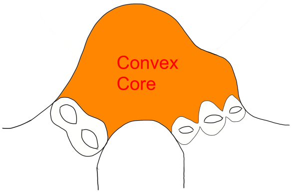
Geodesic planes in M^*

Definition

A geodesic plane in M^* is a non-empty intersection

$$P^* := P \cap M^*.$$

- P^* is connected.



Question

- Can we classify possible closures of P^* in M^*?
- Are all closures submanifolds of M^*?

Yes for convex cocompact acylindrical manifolds.

No in general.

Question

- Can we classify possible closures of P^* in M^*?
- Are all closures submanifolds of M^*?

Yes for convex cocompact acylindrical manifolds.

No in general.

Convex cocompact Acylindrical manifolds

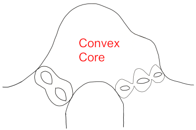
M convex cocompact, non-fuchsian

Definition

M is acylindrical if

$$\text{core}(M)$$

- has incompressible boundary, and
- has no essential cylinders.



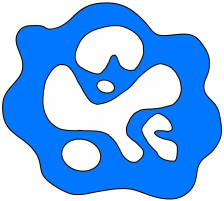
Definition

A convex cocompact M is acylindrical if Λ is a Sierpinski curve.

A cpt $\Lambda \subset S^2$ is a Sierpinski curve if

$$S^2 - \Lambda = \bigcup B_i$$

is a dense union of Jordan disks B_i with mutually disj. closures and $\text{diam}(B_i) \to 0$.



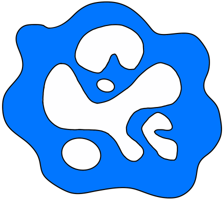
Definition

A convex cocompact M is acylindrical if Λ is a Sierpinski curve.

A cpt $\Lambda \subset S^2$ is a Sierpinski curve if

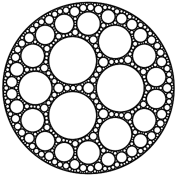
$$S^2 - \Lambda = \bigcup B_i$$

is a dense union of Jordan disks B_i with mutually disj. closures and $\mathrm{diam}(B_i) \to 0$.



Convex cocompact rigid Acylindrical manifolds

If M is convex cocompact s.t. core(M) has totally geodesic bdry, M is rigid acylindrical.



Such M are called rigid as the double of core(M) is a closed 3-mfld, obeying Mostow rigidity.

Class of convex cocompact acylindrical mflds

- The acylindrical condition on a CC 3-mfld M depends only on the topology of M.
- Any hyp. 3-mfld quasi-isometric to a CC acy one is CC acylindrical.
- $\mathrm{QI}(\Gamma\backslash\mathbb{H}^3) = \prod_i \mathrm{Teich}(S_i)$ where $\partial(\mathrm{core}(M)) = \bigcup_i S_i$.

Closed-Dense dichotomy

Let M be a convex cocompact acylindrical mfld.

Theorem 1 (McMullen-Mohammadi-O.)

Any geodesic plane $P^* \subset M^*$

- *closed; or*
- *dense.*

Moreover, a closed P^* is properly immersed and has non-elementary π_1.

When $\mathrm{Vol}(M) < \infty$, $M^* = M$ and so this is a generalization of Ratner-Shah theorem.

Closed-Dense dichotomy

Let M be a convex cocompact acylindrical mfld.

Theorem 1 (McMullen-Mohammadi-O.)

Any geodesic plane $P^* \subset M^*$

- *closed; or*
- *dense.*

Moreover, a closed P^* is properly immersed and has non-elementary π_1.

When Vol(M) $< \infty$, $M^* = M$ and so this is a generalization of Ratner-Shah theorem.

Properly immersed geodesic planes

Theorem 2 (MMO)

There are at most countably many closed geod. planes in M^*.

- Except for finitely many, all corresponding planes P have infinite area (when $\mathrm{Vol}(M) = \infty$).

Topological equidistribution

Theorem 3 (MMO)

If $P_i^* \subset M^*$ is an inf. seq. of distinct closed planes,

$$\lim P_i^* = M^*$$

in the Hausdorff topology of closed subsets of M^*.

The acylindrical condition is necessary, since these theorems are false in general for a cylindrical manifold.

Topological equidistribution

Theorem 3 (MMO)

If $P_i^* \subset M^*$ is an inf. seq. of distinct closed planes,

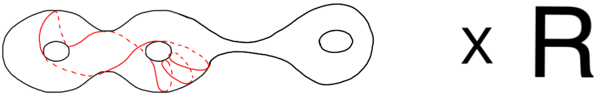
$$\lim P_i^* = M^*$$

in the Hausdorff topology of closed subsets of M^*.

The acylindrical condition is necessary, since these theorems are false in general for a cylindrical manifold.

Counterexamples: Cylindrical manifolds

Consider a fuchsian 3-manifold $M \simeq S \times \mathbb{R}$.

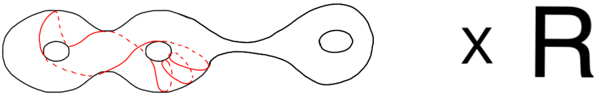


If $\gamma \subset S$ is a geodesic and $P \perp S$ with $P \cap S = \gamma$, then

$$\overline{P} \simeq \overline{\gamma} \times \mathbb{R}.$$

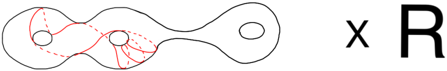
Counterexamples: Cylindrical manifolds

Consider a fuchsian 3-manifold $M \simeq S \times \mathbb{R}$.



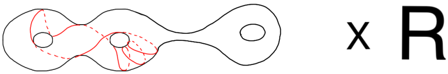
If $\gamma \subset S$ is a geodesic and $P \perp S$ with $P \cap S = \gamma$, then

$$\overline{P} \simeq \overline{\gamma} \times \mathbb{R}.$$



- If $\overline{\gamma}$ is wild, so is $\overline{P} \simeq \overline{\gamma} \times \mathbb{R}$.
- If γ is closed, so is P.
- As $\exists$ uncountably many P s.t. $P \cap S = \gamma$, $\exists$ uncountably many closed planes.

For $M = S \times \mathbb{R}$ fuchsian, core$(M) = S$; $M^* = \emptyset$.



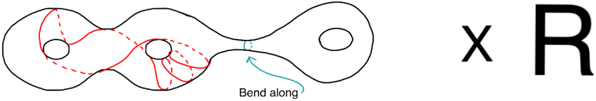
- If $\overline{\gamma}$ is wild, so is $\overline{P} \simeq \overline{\gamma} \times \mathbb{R}$.
- If γ is closed, so is P.
- As $\exists$ uncountably many P s.t. $P \cap S = \gamma$, $\exists$ uncountably many closed planes.

For $M = S \times \mathbb{R}$ fuchsian, core$(M) = S$; $M^* = \emptyset$.

Counterexamples:Quasi-fuchsian mflds

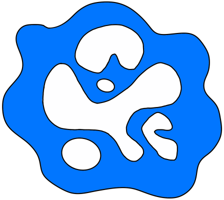
By bending along a separating simple closed geodesic γ_0, we get a quasi-fuchsian mfld M with $M^* \neq \emptyset$ s.t. if γ is far enough from γ_0,

$$\overline{P} = \overline{\gamma} \times \mathbb{R}$$



From now on, assume that

$$M = \Gamma\backslash\mathbb{H}^3 : \text{convex cocompact, acylindrical}$$



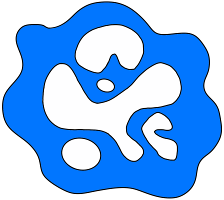
In order to study the closure of a plane in M, we lift this problem to the frame bundle

$$\mathrm{F}(M)$$

which is a homogeneous space $\Gamma\backslash \mathrm{PSL}_2(\mathbb{C})$ and study homogeneous dynamics on it.

From now on, assume that

$M = \Gamma\backslash\mathbb{H}^3$: convex cocompact, acylindrical



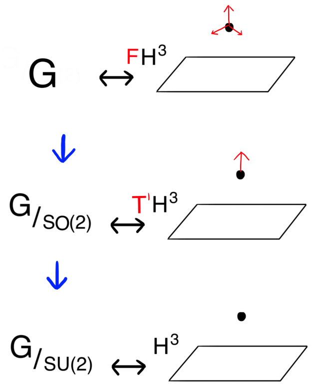
In order to study the closure of a plane in M, we lift this problem to the frame bundle

$$\mathrm{F}(M)$$

which is a homogeneous space $\Gamma\backslash\,\mathrm{PSL}_2(\mathbb{C})$ and study homogeneous dynamics on it.

$PSL_2(\mathbb{C})$= Frame bundle $F(\mathbb{H}^3)$

$g \leftrightarrow (e_1, e_2, e_3)$



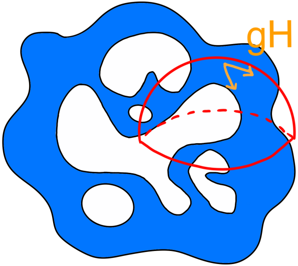
$$F(M) = \Gamma \backslash F(\mathbb{H}^3) = \Gamma \backslash G$$

$\mathrm{PSL}_2(\mathbb{R})$-orbit closure

- $G := \mathrm{PSL}_2(\mathbb{C})$, $H := \mathrm{PSL}_2(\mathbb{R})$.

$$\begin{array}{ccc} \mathrm{F}(M) = \Gamma\backslash G & \supset & xH \\ \downarrow \pi & & \downarrow \\ M = \Gamma\backslash\mathbb{H}^3 & \supset & \mathrm{P} \end{array}$$

Classification of possible closures of $P^* \subset M^*$ follows from classification of H-orbit closures in $\Gamma\backslash G$.

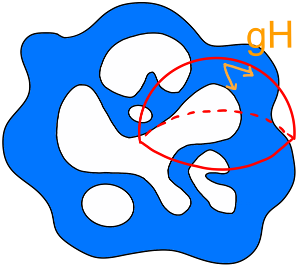


$\mathrm{PSL}_2(\mathbb{R})$-orbit closure

- $G := \mathrm{PSL}_2(\mathbb{C})$, $H := \mathrm{PSL}_2(\mathbb{R})$.

$$\begin{array}{ccc} \mathrm{F}(M) = \Gamma\backslash G & \supset & xH \\ \downarrow \pi & & \downarrow \\ M = \Gamma\backslash\mathbb{H}^3 & \supset & \mathrm{P} \end{array}$$

Classification of
possible closures of $P^* \subset M^*$
follows from classification
of H-orbit closures in $\Gamma\backslash G$.



H-orbit closure theorem

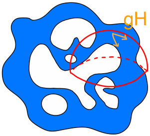
Let F^* be the minimal H-inv (open) subset above M^*, i.e.,

$$F^* := \bigcup \{xH : \pi(xH) \cap M^* \neq \emptyset\} \subset \Gamma \backslash G.$$

Note if $\mathrm{Vol}(M) < \infty$, then $F^* = \Gamma \backslash G$.

Theorem

$\forall x \in F^*$, xH is closed or dense in F^*;



H-orbit closure theorem

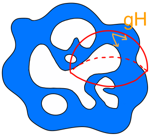
Let F^* be the minimal H-inv (open) subset above M^*, i.e.,

$$F^* := \bigcup \{xH : \pi(xH) \cap M^* \neq \emptyset\} \subset \Gamma \backslash G.$$

Note if $\mathrm{Vol}(M) < \infty$, then $F^* = \Gamma \backslash G$.

Theorem

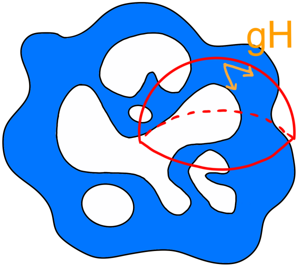
$\forall x \in F^*$, xH is closed or dense in F^*;



Description of H-orbit closures in $\Gamma\backslash G$

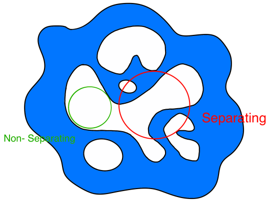
$= $ Description of Γ-orbit closures in G/H

$G/H =$ space of all oriented circles in S^2 :=$\mathcal{C}$



Γ-orbit closure theorem

$$\mathcal{C}^* := \{C \in \mathcal{C} : C \text{ separates } \Lambda\}$$

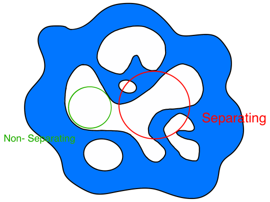


Theorem

$\forall C \in \mathcal{C}^*$, ΓC *is closed or dense in* $\mathcal{C}^*$.

Γ-orbit closure theorem

$$\mathcal{C}^* := \{C \in \mathcal{C} : C \text{ separates } \Lambda\}$$



Theorem

$\forall C \in \mathcal{C}^*$, ΓC is closed or dense in $\mathcal{C}^*$.

Scarcity of recurrence of unipotent flow

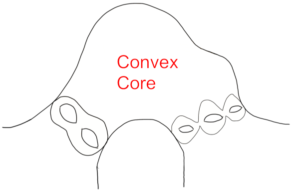
Our approach is based on homogeneous dynamics, more precisely, unipotent dynamics in $\Gamma\backslash G$ for the action of the unipotent group:

$$U := \{u_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

The main difficulty in carrying out unipotent dynamics in the infinite volume setting is the lack of recurrence of unipotent flows.

When $\operatorname{Vol}\Gamma\backslash G = \infty$, for almost all x and for any compact $\Omega \subset \Gamma\backslash G$,

$$\limsup \frac{1}{T}\ell\{t \in [0, T] : xu_t \in \Omega\} = 0.$$



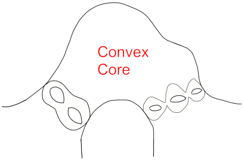
K-thick recurrence in the acylindrical case

We construct an A-inv. compact subset $\Omega \subset \Gamma \backslash G$ and $K > 1$ s.t.

- $\forall x \in \Omega$, the orbit xU has the K-thick recurrence to Ω, that is, $\forall T > 0$,

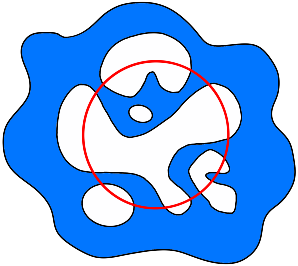
$$xu_t \in \Omega \quad \text{for some } t \in [-KT, -T] \cup [T, KT];$$

- $F^* \subset \Omega H$



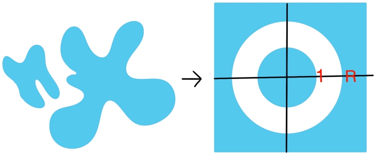
Construction of such a compact subset $\Omega \subset \Gamma\backslash G$ is directly related to the study of

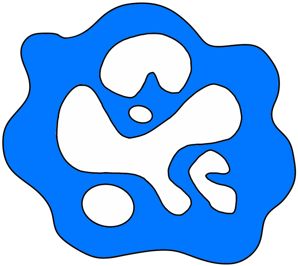
- the structure of Λ
- its circular slices



Theorem A

The limit set of a CC acylindrical gp is a Sierpinski curve of positive modulus.





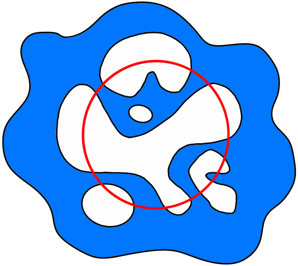
For $S^2 - \Lambda = \bigcup B_i$, there is a unif. lower bdd for the modular distances among B_i's:

$$\mathrm{mod}(\Lambda) := \inf_{i \neq j} \mathrm{mod}(S^2 - (\overline{B_i} \cup \overline{B_j})) > 0$$

Circular slices of a Sierpinski curve

Theorem B

Let Λ be a Sierpinski curve of positive modulus. Then for any C separating Λ, $C \cap \Lambda$ contains a Cantor set T_C of modulus $\epsilon > 0$.



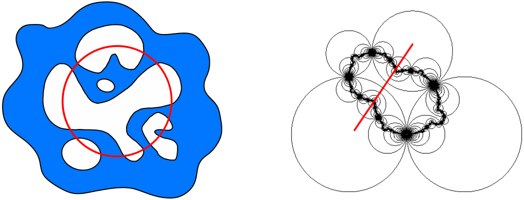
For $C - T_C = \bigcup I_i$,

$$\mathrm{mod}(T_C) := \inf_{i \neq j} \mathrm{mod}(S^2 - (\overline{I_i} \cup \overline{I_j})) \geq \epsilon$$

Circular slices of the limit set

Theorem

Let Γ be a CC acylindrical gp. For any circle C separating Λ, $C \cap \Lambda$ contains a Cantor subset T_C of modulus $\epsilon > 0$.



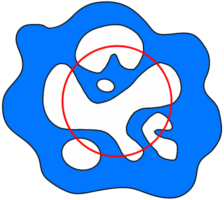
$$\Omega = \Gamma \backslash \{\text{geodesics connecting pts in } \cup_{C \in \mathcal{C}} T_C \subset \Lambda\}$$

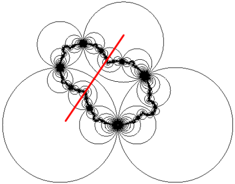
has the desired K-thick recurrence property for unipotent flows

Circular slices of the limit set

Theorem

Let Γ be a CC acylindrical gp. For any circle C separating Λ, $C \cap \Lambda$ contains a Cantor subset T_C of modulus $\epsilon > 0$.





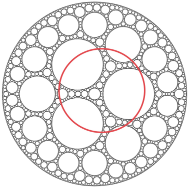
$$\Omega = \Gamma \backslash \{\text{geodesics connecting pts in } \cup_{C \in \mathcal{C}} T_C \subset \Lambda\}$$

has the desired K-thick recurrence property for unipotent flows

Rigid case

Theorem

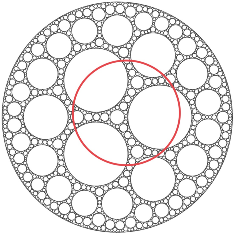
For any $C \in \mathcal{C}^*$, $T_C = C \cap \Lambda$ is a Cantor set of modulus $\epsilon > 0$.



If $C - \Lambda = \bigcup I_i$ with $I_i \subset B_i$,

$d(\text{hull}(I_i), \text{hull}(I_j)) \geq d(\text{hull}(B_i), \text{hull}(B_j)) \geq \text{sys}(\text{Double of core}(M))/2$

Rigid case



So the renormalized frame bundle of M

$$\text{RFM} = \Gamma \backslash \{\text{geod. connecting pts in } \Lambda\} \subset \Gamma \backslash G$$

has the desired recurrence property for unipotent flows.

Geometrically finite acylindrical mfld

Any hyperbolic 3-mfld M with finitely generated $\pi_1(M)$ has a compact conn. submanifold N, called Scott core, s.t. the inclusion $N \subset M$ induces an isomorphism $\pi_1(N) \simeq \pi_1(M)$.

Definition

A geometrically finite M is acylindrical if

$$\text{Scott-core}(M)$$

- has incompressible boundary, and
- has no essential cylinders.

Theorem (Benoist-O.)

Let M be geometrically finite and acylindrical. Then

- *Any geodesic plane P^* in M^* is closed or dense.*
- *There are only countably many closed P^*.*
- *Any inf. seq of distinct closed P_i^* become dense in M^*.*

Circular slices of the limit set

Theorem

Let Γ be a geometrically finite acylindrical gp. For any circle C separating Λ, $C \cap \Lambda$ contains a Cantor subset of modulus $\epsilon > 0$.

In this case, Λ is a quotient of a Sierpinksi curve of positive modulus:

$$S^2 - \Lambda = \bigcup T_\ell$$

where T_ℓ's are maximal trees of disks so that the modular distance between $\overline{T_\ell}$ and $\overline{T_k}$ are uniformly bounded from below for all $\ell \neq k$

Circular slices of the limit set

Theorem

Let Γ be a geometrically finite acylindrical gp. For any circle C separating Λ, $C \cap \Lambda$ contains a Cantor subset of modulus $\epsilon > 0$.

In this case, Λ is a quotient of a Sierpinksi curve of positive modulus:

$$S^2 - \Lambda = \bigcup T_\ell$$

where T_ℓ's are maximal trees of disks so that the modular distance between $\overline{T_\ell}$ and $\overline{T_k}$ are uniformly bounded from below for all $\ell \neq k$

Planes and Coverings

Let

$$p : M \to M_0 \text{ be a covering map}$$

where M is a GF acy hyp 3-mfld and M_0 has finite vol.

Theorem (McMullen-Mohammadi-O., Benoist-O.)

Let M_0 be arithmetic. For a geod. plane $P \subset M$ with $P^* \neq \emptyset$,

- *P^* is closed in M^* iff $p(P)$ is closed in M_0;*
- *P^* is dense in M^* iff $p(P)$ is dense in M_0.*

Theorem (Benoist -O.)

$\exists$ a non-arith. mfld M_0 covered by M and a geod. plane $P \subset M$ s.t P^* is closed in M^* and $p(P)$ is dense in M_0.

Planes and Coverings

Let

$$p : M \to M_0 \text{ be a covering map}$$

where M is a GF acy hyp 3-mfld and M_0 has finite vol.

Theorem (McMullen-Mohammadi-O., Benoist-O.)

Let M_0 be arithmetic. For a geod. plane $P \subset M$ with $P^* \neq \emptyset$,

- *P^* is closed in M^* iff $p(P)$ is closed in M_0;*
- *P^* is dense in M^* iff $p(P)$ is dense in M_0.*

Theorem (Benoist -O.)

$\exists$ a non-arith. mfld M_0 covered by M and a geod. plane $P \subset M$ s.t P^* is closed in M^* and $p(P)$ is dense in M_0.

Planes and Coverings

Let

$$p : M \to M_0 \text{ be a covering map}$$

where M is a GF acy hyp 3-mfld and M_0 has finite vol.

Theorem (McMullen-Mohammadi-O., Benoist-O.)

Let M_0 be arithmetic. For a geod. plane $P \subset M$ with $P^* \neq \emptyset$,

- *P^* is closed in M^* iff $p(P)$ is closed in M_0;*
- *P^* is dense in M^* iff $p(P)$ is dense in M_0.*

Theorem (Benoist -O.)

$\exists$ a non-arith. mfld M_0 covered by M and a geod. plane $P \subset M$ s.t P^* is closed in M^* and $p(P)$ is dense in M_0.