Bottom of the L^2-spectrum

Hee Oh

(Yale University)

May 2023

G conn. semi-simple real alg gp

(e.g $SL_n\mathbb{R}$, $SO(p,q)$, $\cdots$)

$X = G/K$ Riem. sym space

$M = \Gamma\backslash X$ $\quad \Gamma < G$ discrete

$L^2(M) \supset W^1(M)$ closure of $C_c^\infty(M)$

w.r.t $\|f\|_{W^1}^2 = \|f\|_2^2 + \|\mathrm{grad} f\|_2^2$

Δ = Laplace operator

uniq self-adjoint ext of

$\Delta = \mathrm{div} \circ \mathrm{grad}$ on $C_c^\infty(M)$ to $W^1(M)$

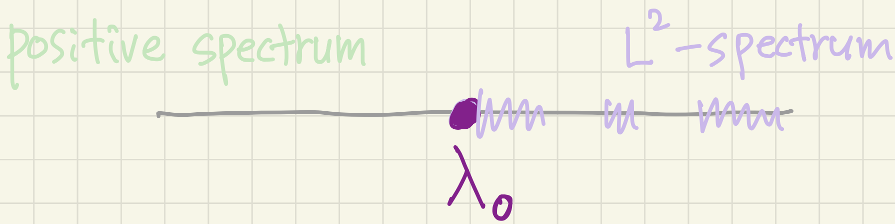
$\sigma(M) = L^2$-spectrum of $-\Delta \subset [0,\infty)$

$= \{ \lambda \in \mathbb{C} \mid \Delta + \lambda$ does not have

a bdd inverse $(\Delta+\lambda)^{-1} : L^2(M) \to W^1(M) \}$

$$\lambda_0 = \lambda_0(M) = \text{the bottom of } \sigma(M)$$

$$= \inf \left\{ \frac{\int_M \|\mathrm{grad} f\|^2}{\int_M |f|^2} \;\middle|\; f \in C_c^\infty(M) \right\}$$



Sullivan (1979)

① $\lambda_0 \in \sigma(M) \subset [\lambda_0, \infty)$

② positive spectrum $\subset (-\infty, \lambda_0]$

Question : Is λ_0 an atom of the L^2-spectral measure ?

I.e. $\exists\ \phi_0 \in L^2(M)$ s.t $-\Delta \phi_0 = \lambda_0 \phi_0$?

necessarily positive

- If $\Gamma < G$ lattice, i.e., $\mathrm{Vol}(M) < \infty$,

  YES $\lambda_0 = 0$ & $1_M \in L^2(M)$

- If $\Gamma = \{e\}$, **NO**

  Harish-Chandra $\lambda_0 = \|\rho\|^2$ half sum of pos. roots

  $\sigma(M) = [\|\rho\|^2, \infty)$

Discrete & Non-lattice subgps?

<u>Example</u> Hitchin subgp

$\pi_n : PSL_2\mathbb{R} \longrightarrow PSL_n\mathbb{R}$ irred rep

$\Gamma_0 < PSL_2\mathbb{R}$ uniform lattice

$\pi_n|_{\Gamma_0} \in \mathrm{Rep}(\Gamma, PSL_n\mathbb{R})/\sim$

$\sigma : \Gamma_0 \longrightarrow PSL_n\mathbb{R}$ Hitchin

if $\sigma \in$ Conn. component of $\pi_n|_{\Gamma_0}$

$\cong \mathbb{R}^{(2g-2)(n^2-1)}$

Hitchin $g = \text{genus}(\Gamma_0 \backslash \mathbb{H}^2)$

Labourie (~2005)
Fock-Goncharov

σ discrete & faithful

$\Gamma = \sigma(\Gamma_0) < PSL_n\mathbb{R}$

Hitchin subgp, $\mathrm{Vol}(\Gamma \backslash PSL_n\mathbb{R}) = \infty$

Thm (Edwards-O. 2022)

$\Gamma < PSL_n\mathbb{R}$ Hitchin $n \geq 3$

$\Rightarrow$ (1) $\lambda_0 = \|\rho\|^2$, $\sigma(M) = [\|\rho\|^2, \infty)$

(2) λ_0 is not an atom for the spectral measure

(2) is true for any Anosov subgp in rank $G \geq 2$.

Characterization of a higher rank lattice

Thm (Edwards-Fraczyk-Lee-O.)

$\Gamma < G$ simple & rank ≥ 2

Zariski dense

More generally, G : s.s with no rank 1 factors

$\mathrm{vol}(\Gamma \backslash G) < \infty$

iff λ_0 is an atom for the spectral meas.

If $\mathrm{vol}(M) = \infty$, No positive L^2-eigenftn on M.

This is in stark contrast to rank 1 situation

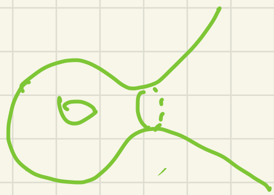
$X = \mathbb{H}^n \quad G = SO^\circ(n,1) \qquad \Gamma < G$ Z.d

δ_Γ = the critical exp of Γ

$= \text{abs. conv of } s \longmapsto \sum_{\gamma\in\Gamma} e^{-s\,d(o,\gamma o)}$

$0 < \delta_\Gamma \le n-1$

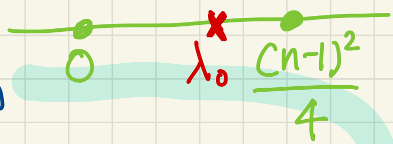
Γ : geometrically finite



($\Gamma \backslash \mathbb{H}^n$ has finite-sided polyhedron fund. dom)

If $n=2$, Γ : g.f $=$ Γ is f.g

Elstrodt, Patterson, Sullivan



$\delta_\Gamma > \frac{n-1}{2}$ iff λ_0 is an atom.

$\lambda_0 = \delta(n-1-\delta)$

$\left(\Leftrightarrow \; 0 \le \lambda_0 < \frac{(n-1)^2}{4}\right)$

true in general rank one : Hamenstadt

Characterization of a higher rank lattice

Thm (Edwards - Fraczyk - Lee - O.)

$\Gamma < G$ simple & rank ≥ 2

Zariski dense

More generally,

G : s.s with no rank 1 factors

$\mathrm{Vol}(\Gamma \backslash G) < \infty$

iff λ_0 is an atom for the spectral meas.

G simple higher-rank

Questions

$\Gamma < G$ Z. dense, $\mathrm{vol}(\Gamma\backslash G) = \infty$

(1) Can there be any L^2- Laplace eigenftn ?

(not nece. positive)

(2) $\exists\ \Gamma$ s.t $L^2(\Gamma\backslash G)$ is non-tempered?

$L^2(\Gamma\backslash G)$ tempered

$\Leftrightarrow$ $L^2(\Gamma\backslash G) \propto L^2(G)$

$\Leftrightarrow$ All matrix coefficients are in $L^{2+\varepsilon}(G)$ $\forall \varepsilon > 0$.

For $\Gamma < PSL_n\mathbb{R}$ $n \geq 3$,
Hitchin

$L^2(\Gamma\backslash G)$ is tempered.

(Edwards-O.)

Γ- conformal densities on $\mathcal{F} = G/P$

$P = MAN$ minimal para.

Patterson, Sullivan, Quint

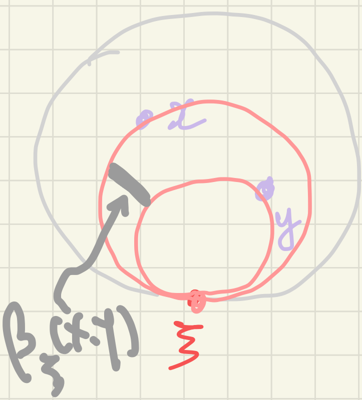
$\psi \in \mathfrak{a}^*$

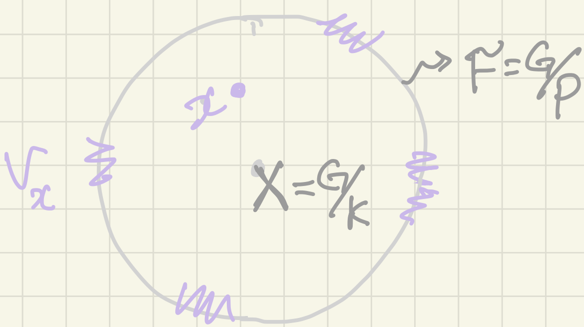
finite Borel measure on $\mathcal{F}$

A family $\nu = \{ \nu_x \mid x \in X = G/K \}$ is a (Γ, ψ) - conformal density

if $$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\psi(\beta_\xi(x,y))} \quad \forall x, y \in X$$

$$\gamma_* \nu_x = \nu_{\gamma(x)} \quad \forall \gamma \in \Gamma$$





Γ-conformal density $\nu = \{\nu_x \mid x \in X\}$

→ pos. eigenftn on $M = \Gamma \backslash X$

$$E_\nu(x) = |\nu_x| = \nu_x(\tilde{F})$$

$\boxed{A}$

Thm (Edwards-O.)

Any positive L^2 - eigenftn is of the form E_ν for some Γ- conf. density ν

based on Sullivan's thm:

$\exists$ at most one positive L^2-ef

(up to a const. multiple).

Higher rank Bowen-Margulis-Sullivan measure.

For a pair

$$\begin{cases} \nu_1 : (\Gamma, \psi_1) \text{ - conformal density} \\ \nu_2 : (\Gamma, \psi_2) \text{ - conformal density} \end{cases}$$

$\rightsquigarrow$ $\mathbf{m}_{\nu_1,\nu_2}$ A (quasi)-inv measure on $\Gamma\backslash G$

Hopf Parametrization

$$P \cap P^{+} = AM$$

$$G/M \simeq G/P^{+} \times' G/P \times \mathfrak{a}$$

$$g \longrightarrow (\underset{\overset{\shortparallel}{g^{+}}}{gP^{+}},\ \underset{\overset{\shortparallel}{g^{-}}}{gP},\ \underset{\overset{\shortparallel}{b}}{\beta_{gP}(o, go)})$$

$$dm_{\nu_1,\nu_2}(g) \simeq e^{\psi_1(\beta_{g^+}(o,go)) + \psi_2(\beta_{g^-}(o,go))} d\nu_{1,o}(g^{+})\, d\nu_{2,o}(g^{-})\, db$$

left Γ-inv & right A quasi-inv.

$\rightsquigarrow$ BMS measure on $\Gamma \backslash G/M$

$\boxed{B}$ Finite BMS $\Rightarrow$ Finite Haar.

in higher-rank

Thm (Fraczyk - Lee 2023)

$\Gamma < G$ Z. dense $\quad \mathrm{Vol}(\Gamma \backslash G) = \infty$

$\uparrow$

no rank 1 factors

$$\Rightarrow \quad m_{\nu_1, \nu_2}(\Gamma \backslash G) = \infty$$

Uses high-entropy methods due to Katok - Einsiedler - Lindenstrauss.

C

Smearing thm Thurston-Sullivan
Edwards-O.

$$m_{v_1, v_2}(\Gamma \backslash G) \ll \int_M E_{v_1} \cdot E_{v_2}$$

In particular, if $|m_{v,v}| = \infty$,
$E_v \notin L^2(M)$

Thank

you !