AHLFORS REGULARITY OF PATTERSON-SULLIVAN MEASURES OF ANOSOV GROUPS AND APPLICATIONS

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Abstract. For all Zarski dense Anosov subgroups of a semisimple real algebraic group, we prove that their limit sets are Ahlfors regular for intrinsic conformal premetrics. As a consequence, we obtain that a Patterson-Sullivan measure is equal to the Hausdorff measure if and only if the associated linear form is symmetric. We also discuss several applications, including analyticity of (p, q)-Hausdorff dimensions on the Teichmüller spaces, new upper bounds on the growth indicator, and L^2-spectral properties of associated locally symmetric manifolds.

1. Introduction

Let G be a connected semisimple real algebraic group. Let $\Gamma < G$ be a discrete subgroup. Patterson-Sullivan measures are certain families of Borel measures on a generalized flag variety, supported on the limit set of Γ. They play a crucial role in the study of dynamics on the associated locally symmetric space, especially in the counting and equidistribution of Γ-orbits of various geometric objects. The original construction is due to Patterson and Sullivan for Kleinian groups ([44], [55]), which was generalized by Quint [49] (see [2] and [13] for earlier works).

Sullivan showed that for convex cocompact Kleinian groups of $\text{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$, Patterson-Sullivan measures are Ahlfors regular Hausdorff measures on the limit sets in $\mathbb{S}^{n-1}$ [55, Theorem 8]. Since Patterson-Sullivan measures are constructed from the weighted Dirac measures on an orbit of Γ in the symmetric space $\mathbb{H}^n_{\mathbb{R}}$, it is remarkable that they can be given the geometric characterization purely in terms of the internal metric on the limit set of Γ which is a subset of the boundary $\partial\mathbb{H}^n_{\mathbb{R}} \simeq \mathbb{S}^{n-1}$.

In recent decades, Anosov subgroups have emerged as a higher rank generalization of convex cocompact Kleinian groups. Therefore it is natural to ask when the Patterson-Sullivan measures of Anosov subgroups arise as Ahlfors regular Hausdorff measures on the limit sets with respect to appropriate metrics. The main goal of this paper is to answer this question.

To state our results, fix a Cartan decomposition $G = KA^+K$ where K is a maximal compact subgroup and A^+ is a positive Weyl chamber of a maximal real split torus A. We denote by X the associated Riemannian symmetric space G/K. Let $\mathfrak{g}$ and $\mathfrak{a}$ denote the Lie algebras of G and A

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respectively, and set $\mathfrak{a}^+ = \log A^+$. Let Π denote the set of all simple roots of $(\mathfrak{g}, \mathfrak{a})$ with respect to the choice of $\mathfrak{a}^+$.

Fix a non-empty subset θ of Π. Let P_θ be the standard parabolic subgroup of G associated with θ. The quotient space

$$\mathcal{F}_\theta = G/P_\theta$$

is called the θ-boundary, or a generalized flag variety. We denote by Λ_θ the limit set of Γ in $\mathcal{F}_\theta$ [3]. For $\theta = \Pi$, we omit the subscript θ from now on; so $P = P_\Pi$ is a minimal parabolic subgroup of G. Set $\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$ and let $\mathfrak{a}_\theta^*$ denote the set of all linear forms on $\mathfrak{a}_\theta$. We may think of $\mathfrak{a}_\theta^*$ as a subspace of $\mathfrak{a}^*$ via the canonical projection $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ (2.3). For $\psi \in \mathfrak{a}_\theta^*$, a (Γ, ψ)-*Patterson-Sullivan measure* is a Borel probability measure ν on Λ_θ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda_\theta$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e, \gamma))}$$

where β denotes the Busemann map (see (2.8)).

A finitely generated subgroup $\Gamma < G$ is called θ-*Anosov* if there exists a constant $C > 1$ such that for all $\alpha \in \theta$,

$$\alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C \quad \text{for all } \gamma \in \Gamma$$

where $|\cdot|$ is a word metric on Γ with respect to a fixed finite generating set and $\mu : G \to \mathfrak{a}^+$ is the Cartan projection defined by the condition that $g \in K(\exp \mu(g))K$ for all $g \in G$. See ([34], [25], [28], [29], [24], [7], etc.) for other equivalent definitions of Anosov subgroups.

In the rest of the introduction, let Γ be a non-elementary[1] θ-Anosov subgroup of G. The space of all Patterson-Sullivan measures of Γ is parameterized by the set $\mathscr{T}_\Gamma \subset \mathfrak{a}_\theta^*$ of all linear forms tangent to the θ-growth indicator ψ_Γ^θ (Definition 3.1):

$$\mathscr{T}_\Gamma = \{\psi \in \mathfrak{a}_\theta^* : \ \psi \geq \psi_\Gamma^\theta, \ \psi(u) = \psi_\Gamma^\theta(u) \quad \text{for some } u \in \mathfrak{a}_\theta - \{0\}\}.$$

More precisely, for any $\psi \in \mathscr{T}_\Gamma$, there exists a unique (Γ, ψ)-Patterson-Sullivan measure

$$\nu_\psi$$

and every Patterson-Sullivan measure of Γ arises in this way (Theorem 3.4). Denote by $\mathcal{L}_\theta \subset \mathfrak{a}_\theta^+$ the θ-limit cone of Γ, which is the asymptotic cone of $p_\theta(\mu(\Gamma))$. Then $\mathscr{T}_\Gamma$ is in bijection with the set $\{\psi \in \mathfrak{a}_\theta^* : \psi > 0 \text{ on } \mathcal{L}_\theta - \{0\}\}/\sim$, where $\psi_1 \sim \psi_2$ if and only if $\psi_1 = c \cdot \psi_2$ for some $c > 0$. When the limit cone $\mathcal{L}_\theta$ has non-empty interior (e.g., when Γ is Zariski dense in G), $\mathscr{T}_\Gamma$ is homeomorphic to $\mathbb{R}^{\#\theta - 1}$.

[1]the limit set Λ_θ has at least three points.

Ahlfors regularity and Hausdorff measures. Anosov subgroups of a rank one Lie group G are precisely convex cocompact subgroups. In general rank one groups, the unique Patterson-Sullivan measure of Γ is Ahlfors regular and coincides with the Hausdorff measure on Λ with respect to a K-invariant *sub-Riemannian* metric on the boundary $\partial_\infty X$ which is defined in terms of the Gromov product [17, Theorem 5.4]. Except for the case of $\mathrm{SO}(n, 1)$, this sub-Riemannian metric is not a Riemannian metric.

In this paper, we prove an analogous theorem for a general Anosov subgroup. Let $\psi \in \mathscr{T}_\Gamma$. The θ-Anosov property of Γ implies that any two distinct points of Λ_θ are in general position and hence the following defines a premetric[2] on Λ_θ: for $\xi, \eta \in \Lambda_\theta$,

$$d_\psi(\xi, \eta) = \begin{cases} e^{-\psi(\mathcal{G}(\xi,\eta))} & \text{if } \xi \neq \eta \\ 0 & \text{if } \xi = \eta \end{cases} \tag{1.1}$$

where $\mathcal{G}$ is the $\mathfrak{a}$-valued Gromov product (see Definition 2.3). This premetric turns out to be a correct replacement of the sub-Riemannian metric of the rank one case.

For $s > 0$, we denote by $\mathcal{H}_\psi^s$ the s-dimensional Hausdorff measure on Λ_θ with respect to the premetric d_ψ, which is a Borel outer measure (9.1). We write $\mathcal{H}_\psi$ for $\mathcal{H}_\psi^1$. It turns out that the metric properties of the Patterson-Sullivan measure ν_ψ depends on the symmetricity of $\psi \in \mathfrak{a}_\theta^*$: ψ is called *symmetric* if ψ is invariant under the opposition involution i of $\mathfrak{a}$ (see (2.2)).

Our main theorem is as follows:

Theorem 1.1. *Let Γ be a non-elementary θ-Anosov subgroup of G. Let $\psi \in \mathscr{T}_\Gamma$ be a symmetric linear form. The Patterson-Sullivan measure ν_ψ is Ahlfors 1-regular and equal to the one-dimensional Hausdorff measure $\mathcal{H}_\psi$, up to a constant multiple.*

The Ahlfors 1-regularity of ν_ψ means that there exists $C \geq 1$ such that for any $\xi \in \Lambda_\theta$ and $0 \leq r < \operatorname{diameter}(\Lambda_\theta, d_\psi)$,

$$C^{-1} r \leq \nu_\psi(B_\psi(\xi, r)) \leq C r \tag{1.2}$$

where $B_\psi(\xi, r) = \{\eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r\}$ (see Definition 8.1). Noting that $\mathcal{H}_{s\psi} = \mathcal{H}_\psi^s$ for $s > 0$, the reason that the Patterson-Sullivan measure is the *one-dimensional* Hausdorff measure in the above theorem is due to the normalization of ψ made by the choice that ψ is a tangent form, i.e., $\psi \in \mathscr{T}_\Gamma$ (see Remark 8.3).

Remark 1.2. If ψ has gradient in the interior of $\mathfrak{a}_\theta^+$, then ψ can be used to define a Finsler metric on X and Dey-Kapovich [18, Theorem A] showed that ν_ψ is the Hausdorff measure, without addressing the Ahlfors regularity (see Remark 9.4). Note that Hausdorff measures need not be Ahlfors-regular

[2]On a topological space X, a premetric d is a positive definite continuous function $d : X \times X \to \mathbb{R}$ such that $d(x, x) = 0$ for all $x \in X$.

in general. Our approach in this paper is different; indeed, we first establish the Ahlfors regularity of ν_ψ and deduce the rest as a consequence of this.

The opposition involution i of $\mathfrak{a}$ is known to be trivial if and only if G does not have a simple factor of type A_n ($n \geq 2$), D_{2n+1} ($n \geq 2$) or E_6 [57, 1.5.1]. When i is non-trivial, the symmetric hypothesis on ψ cannot be removed. In fact, we prove the following (Theorem 9.2, see also Remark 9.8):

Theorem 1.3. *Let Γ be a Zariski dense θ-Anosov subgroup of G. For any non-symmetric $\psi \in \mathscr{T}_\Gamma$, ν_ψ is Ahlfors s-regular for some $0 < s < 1$ but is not comparable*[3] *to any $\mathcal{H}^s_\psi$, $s > 0$.*

Critical exponents and Hausdorff dimensions. Denote by $\mathcal{L}_\theta \subset \mathfrak{a}_\theta^+$ the θ-limit cone of Γ, which is the asymptotic cone of $p_\theta(\mu(\Gamma))$. For $\psi \in \mathfrak{a}_\theta^*$ which is positive on $\mathcal{L}_\theta - \{0\}$, we set

$$\delta_\psi := \delta_\psi(\Gamma) = \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma : \psi(\mu(\gamma)) \leq T\}}{T}. \tag{1.3}$$

The Hausdorff dimension of (Λ_θ, d_ψ) is defined as:

$$\dim_\psi \Lambda_\theta := \inf\{s > 0 : \mathcal{H}^s_\psi(\Lambda_\theta) < \infty\}.$$

A natural question is whether $\dim_\psi \Lambda_\theta$ is equal to δ_ψ.

Theorem 1.4. *For any $\psi \in \mathfrak{a}_\theta^*$ which is positive on $\mathcal{L}_\theta - \{0\}$, we have*

$$\dim_\psi \Lambda_\theta = \delta_{\bar\psi}$$

where $\bar\psi = \frac{\psi + \psi\circ \mathrm{i}}{2}$. In particular, if ψ is symmetric, then $\dim_\psi \Lambda_\theta = \delta_\psi$.

Remark 1.5.
- We remark that for ψ non-symmetric, $\dim_\psi \Lambda_\theta$ is not equal to δ_ψ in general (Proposition 9.10).
- When G is of rank-one, Theorem 1.4 is due to Patterson, Sullivan ([44], [55]) and Corlette [17].

Together with a work of Bridgeman-Canary-Labourie-Sambarino [10, Proposition 8.1], Theorem 1.4 implies that for any ψ non-negative on $\mathfrak{a}_\theta^+$, $\dim_\psi \Lambda_\theta$ changes analytically on θ-Anosov representations (Corollary 9.13). We describe one concrete example as follows.

(p, q)-Hausdorff dimension and Teichmüller space. Let Σ be a torsion-free uniform lattice of $\mathrm{PSL}_2(\mathbb{R})$, and let $\mathrm{Teich}(\Sigma)$ be the Teichmüller space:

$$\mathrm{Teich}(\Sigma) = \{\sigma : \Sigma \to \mathrm{PSL}_2(\mathbb{R}) : \text{discrete, faithful representation}\}/\sim$$

where the equivalence relation is given by conjugations. It is well-known that $\mathrm{Teich}(\Sigma) \simeq \mathbb{R}^{6g-6}$ where g is the genus of the surface $\Sigma\backslash\mathbb{H}^2_{\mathbb{R}}$. For $\sigma \in \mathrm{Teich}(\Sigma)$, denote by $\Lambda_\sigma \subset \mathbb{S}^1 \times \mathbb{S}^1$ the limit set of the self-joining subgroup $(\mathrm{id} \times \sigma)(\Sigma) = \{(\gamma, \sigma(\gamma)) : \gamma \in \Sigma\}$, which is well-defined up to translations. The Hausdorff dimension of Λ_σ with respect to a Riemannian metric on $\mathbb{S}^1 \times \mathbb{S}^1$ is equal to 1 for any $\sigma \in \mathrm{Teich}(\Sigma)$ [30, Theorem 1.1]. For

[3] Two measures ν_1 and ν_2 are comparable if $C^{-1}\nu_2 \leq \nu_1 \leq C\nu_2$ for some $C \geq 1$

any pair (p,q) of positive real numbers, consider the premetric on $\mathbb{S}^1 \times \mathbb{S}^1$ given by

$$d_{p,q}(\xi,\eta) = d_{\mathbb{S}^1}(\xi_1,\eta_1)^p d_{\mathbb{S}^1}(\xi_2,\eta_2)^q$$

for any $\xi = (\xi_1,\xi_2)$ and $\eta = (\eta_1,\eta_2)$ in $\mathbb{S}^1 \times \mathbb{S}^1$, where $d_{\mathbb{S}^1}$ is a Riemannian metric on $\mathbb{S}^1$. For a subset $S \subset \mathbb{S}^1 \times \mathbb{S}^1$, denote by $\dim_{p,q} S$ the Hausdorff dimension of S with respect to $d_{p,q}$. Note that on the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$, $d_{p,q}$ is the $(p+q)$-th power of $d_{\mathbb{S}^1}$ and hence

$$\dim_{p,q} \Lambda_{\mathrm{id}} = \frac{1}{p+q}.$$

For each $\sigma \in \mathrm{Teich}(\Sigma)$, denote by $\delta_{p,q}(\sigma)$ the critical exponent of the Poincaré series $s \mapsto \sum_{\gamma\in\Sigma} e^{-s(pd_{\mathbb{H}^2_{\mathbb{R}}}(o,\gamma o)+qd_{\mathbb{H}^2_{\mathbb{R}}}(o,\sigma(\gamma)o))}$.

Corollary 1.6. *Let $p,q>0$.*

(1) *For any $\sigma \in \mathrm{Teich}(\Sigma)$, we have*

$$\dim_{p,q} \Lambda_\sigma = \delta_{p,q}(\sigma);$$

(2) *For any $\sigma \in \mathrm{Teich}(\Sigma)$, we have*

$$\dim_{p,q} \Lambda_\sigma \le \frac{1}{p+q}$$

and the equality holds if and only if $\sigma = \mathrm{id}$;

(3) *The map*

$$\sigma \mapsto \dim_{p,q} \Lambda_\sigma$$

is an analytic function on $\mathrm{Teich}(\Sigma)$.

Part (2) is an immediate consequence of (1) by the rigidity theorem on $\delta_{p,q}(\sigma)$, due to Bishop and Steger [6, Theorem 2] and to Burger [13, Theorem 1(a)]. See Corollary 9.14 for a more general version on convex cocompact representations. If we denote by $f = f_\sigma$ the σ-equivariant homeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$, then $\Lambda_\sigma = \{(x, f(x)) : x \in \mathbb{S}^1\}$ and $\dim_{p,q}\Lambda_\sigma$ can also be understood as the Hausdorff dimension of $\Lambda_\Sigma = \mathbb{S}^1$ with respect to the premetric $d_{\sigma,p,q}(x,y) = d_{\mathbb{S}^1}(x,y)^p d_{\mathbb{S}^1}(f(x),f(y))^q$, $x,y \in \mathbb{S}^1$.

Hausdorff dimension of Λ_θ with respect to a Riemannian metric. We denote by $\dim \Lambda_\theta$ the Hausdorff dimension of Λ_θ with respect to a *Riemannian metric* on $\mathcal{F}_\theta$; since all Riemannian metrics on $\mathcal{F}_\theta$ are Lipschitz equivalent to each other, this is well-defined. With the exception of $G = \mathrm{SO}^\circ(n,1)$, $\dim \Lambda$ is not in general equal to the critical exponent of Γ even in rank one case. See [19] for a discussion on this for the case of $G = \mathrm{SU}(n,1)$.

From Theorem 1.4, we derive an estimate on $\dim \Lambda_\theta$ in terms of critical exponents. Let χ_α denote the Tits weight of G associated to $\alpha \in \Pi$ as given in (10.2). When G is split over $\mathbb{R}$, χ_α is simply the fundamental weight associated to α. We prove:

Theorem 1.7. *For any θ-Anosov subgroup Γ of G, we have*

$$\max_{\alpha\in\theta} \delta_{\chi_\alpha+\chi_{\mathrm{i}(\alpha)}} \leq \dim \Lambda_\theta \leq \max_{\alpha\in\theta} \delta_\alpha.$$

Moreover, both the upper and lower bounds are attained by some Anosov subgroups.

For $G = \mathrm{PSL}_n(\mathbb{R})$, we have the set of simple roots given by

$$\alpha_k(\mathrm{diag}(a_1, \cdots, a_n)) = a_k - a_{k+1}, \quad 1 \leq k \leq n-1.$$

When $G = \mathrm{PSL}_n(\mathbb{R})$ and $\theta = \{\alpha_1\}$, the lower bound in Theorem 1.7 was obtained by Dey-Kapovich [18], and the upper bound by Pozzetti-Sambarino-Wienhard [47] (see also [14]). For some special class of Anosov subgroups, much sharper bounds are known, see ([23], [46], [47], [30]). Recently, Li-Pan-Xu proved that for $G = \mathrm{PSL}_3(\mathbb{R})$, $\dim \Lambda_{\alpha_1}$ coincides with the affinity exponent of Γ [38]. See also ([36], [33]) which show that Λ_θ has Lebesgue measure zero in higher rank and [35] which shows that $\dim \Lambda_\theta$ has a positive co-dimension for all Zariski dense Anosov subgroups of $\mathrm{PSL}_n(\mathbb{R})$, $n \geq 3$.

The novelty of Theorem 1.7 is that it applies to *all* θ-Anosov subgroups of any semisimple real algebraic group. Since both upper and lower bounds are realized by some Anosov subgroups, Theorem 1.7 cannot be improved in this generality. A *Hitchin subgroup* of $\mathrm{PSL}_n(\mathbb{R})$ is the image of a representation $\pi : \Sigma \to \mathrm{PSL}_n(\mathbb{R})$ of a uniform lattice $\Sigma < \mathrm{PSL}_2(\mathbb{R})$ belonging to the same connected component as $\iota|_\Sigma$ in the character variety $\mathrm{Hom}(\Sigma, \mathrm{PSL}_n(\mathbb{R}))/\sim$ where ι is the irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL}_n(\mathbb{R})$ and the equivalence is given by conjugations. Hitchin subgroups are Π-Anosov, as was shown by Labourie [34, Theorem 1.4]. For Hitchin subgroups of $\mathrm{PSL}_n(\mathbb{R})$, we have $\dim \Lambda_\theta = 1$ by ([34], [14, Proposition 1.5]) and $\delta_\alpha = 1$ for all $\alpha \in \Pi$ by [45, Theorem B]. Hence

$$\dim \Lambda_\theta = 1 = \delta_\alpha \quad \text{ for all } \alpha \in \theta.$$

The upper bound in Theorem 1.7 is also obtained for Anosov subgroups of the product of $\mathrm{SO}^\circ(n,1)$'s [30]. For the lower bound, let Γ be the image of a uniform lattice Σ of $\mathrm{PSL}_2(\mathbb{R})$ under the embedding $\mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \begin{pmatrix} \mathrm{PSL}_2(\mathbb{R}) & 0 \\ 0 & I_{n-2} \end{pmatrix} < \mathrm{PSL}_n(\mathbb{R})$ where I_{n-2} is the $(n-2)\times(n-2)$ identity matrix. Then Γ is $\{\alpha_1\}$-Anosov. On one hand, the limit set Λ_{α_1} of Γ in $\mathcal{F}_{\alpha_1} = \mathbb{P}(\mathbb{R}^n)$ is the projective line, and hence $\dim \Lambda_{\alpha_1} = 1$. On the other hand, since $(\chi_{\alpha_1} + \chi_{\mathrm{i}(\alpha_1)})(\mathrm{diag}(a_1, \cdots, a_n)) = a_1 - a_n$, we have

$$\delta_{\chi_{\alpha_1}+\chi_{\mathrm{i}(\alpha_1)}} = \delta_\Sigma = 1 = \dim \Lambda_{\alpha_1}.$$

Therefore the lower bound in Theorem 1.7 is achieved for this example.

Growth indicator bounds and L^2-spectral properties. The growth indicator $\psi_\Gamma = \psi_\Gamma^\Pi : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is a higher rank version of the critical exponent of Γ that captures the growth rate of $\mu(\Gamma)$ in each direction of $\mathfrak{a}$ (Definition 3.1). This was introduced by Quint [48]. Denote by ρ the half-sum of all positive roots of $(\mathfrak{g}, \mathfrak{a})$ counted with multiplicity. Then for any discrete subgroup $\Gamma < G$, $\psi_\Gamma \le 2\rho$, and if G has no rank one factor and $\mathrm{Vol}(\Gamma\backslash G) = \infty$, then Quint showed a gap theorem that $\psi_\Gamma \le 2\rho - \Theta$ where Θ denotes the half-sum of all roots in a maximal strongly orthogonal system of $(\mathfrak{g}, \mathfrak{a})$ ([51], [43], [36, Theorem 7.1]). We obtain the following bound on ψ_Γ for Anosov subgroups:

Corollary 1.8. *For any θ-Anosov subgroup Γ of G, we have*

$$\psi_\Gamma \le \dim \Lambda_\theta \cdot \min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \quad \text{on } \mathfrak{a}. \tag{1.4}$$

Recall that $X = G/K$ denotes the associated Riemannian symmetric space. The size of ψ_Γ is closely related to the spectral properties of the locally symmetric manifold $\Gamma\backslash X$. Let $\lambda_0(\Gamma\backslash X)$ denote the bottom of the L^2-spectrum of $\Gamma\backslash X$ (see (11.4)). As first introduced by Harish-Chandra [26], a unitary representation $(\pi, \mathcal{H}_\pi)$ of G is *tempered* if all of its matrix coefficients belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$, or, equivalently, if π is weakly contained[4] in the regular representation $L^2(G)$. Hence the temperedness of the quasi-regular representation $L^2(\Gamma\backslash G)$ means that $\Gamma\backslash G$ looks like G from the L^2-viewpoints. If a discrete subgroup Γ of G satisfies that $\psi_\Gamma \le \rho$, then $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0(\Gamma\backslash X) = \|\rho\|^2$ as shown in [21, Theorem 1.6] for Π-Anosov groups and in [39] in general. Moreover, $\lambda_0(\Gamma\backslash X)$ is not an L^2-eigenvalue ([21], [20]). However it is not easy to decide whether $\psi_\Gamma \le \rho$ holds or not. We give a criterion on this in terms of $\dim \Lambda_\theta$ using Corollary 1.8.

Define

$$\mathsf{c}_G := \inf\{c > 0 : \sum_{\alpha\in\Pi} \chi_\alpha \le c\cdot\rho\}.$$

If G is $\mathbb{R}$-split, then $\sum_{\alpha\in\Pi}\chi_\alpha = \rho$ [9, Proposition 29], and hence $\mathsf{c}_G = 1$. In general, we have

$$0 < \mathsf{c}_G \le 1 \tag{1.5}$$

by Lemma 10.3 due to Smilga. We obtain:

Corollary 1.9. *Let Γ be a θ-Anosov subgroup. Suppose that*

- $\dim \Lambda_\theta \le \frac{\#\theta}{\mathsf{c}_G}$ *if* $\theta \cap \mathrm{i}(\theta) = \emptyset$;
- $\dim \Lambda_\theta \le \frac{\#\theta}{2\mathsf{c}_G}$ *otherwise.*

Then $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0(\Gamma\backslash X) = \|\rho\|^2$. In particular, the conclusion holds for any Π-Anosov subgroup with $\dim \Lambda \le \frac{\operatorname{rank} G}{2\mathsf{c}_G}$.

See Remark 11.5 for a more general statement. Corollary 1.9 recovers Sullivan's theorem [56] in rank one Lie groups (see Remark 11.6) and immediately applies to many examples of Anosov subgroups with limit sets of low Hausdorff dimensions; for example to all Π-Anosov subgroups of higher

[4] π is weakly contained in a unitary representation σ of G if any diagonal matrix coefficients of π can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of σ.

rank Lie groups with $\dim \Lambda \leq 1$ such as Hitchin subgroups ([14, Proposition 1.5]). Although the conclusion of Corollary 1.9 was already known for Hitchin subgroups by [31] and [21] relying on the work of [45], we obtain a completely different proof in this paper.

Since the opposition involution i of $\mathrm{PSL}_n(\mathbb{R})$ sends the simple root α_i to α_{n-i} for $1 \leq i \leq n-1$, we deduce:

Corollary 1.10. *Let $n \geq 3$. If $\Gamma < \mathrm{PSL}_n(\mathbb{R})$ is $\{\alpha_i\}$-Anosov with $\dim \Lambda_{\alpha_i} \leq 1$ for some $i \neq \frac{n}{2}$, then $L^2(\Gamma\backslash \mathrm{PSL}_n(\mathbb{R}))$ is tempered and $\lambda_0(\Gamma\backslash X) = \|\rho\|^2$.*

This corollary applies to any $(1,1,2)$-hyperconvex subgroup whose Gromov boundary is homeomorphic to a circle, since such a subgroup is $\{\alpha_1\}$-Anosov with $\dim \Lambda_{\alpha_1} = 1$ by Pozzetti-Sambarino-Wienhard [46]. It also applies to the image of a purely hyperbolic Schottky representation of the free group F_k on k-generators in $\mathrm{PSL}_n(\mathbb{R})$ in the sense of Burelle-Treib [12] by [14, Proposition 11.1].

On the proof of Theorem 1.1. The key step is to prove that for a symmetric $\psi \in \mathscr{T}_\Gamma$, the Patterson-Sullivan measure ν_ψ is Ahlfors one-regular. Fix $o = [K] \in X$. The θ-Anosov property of Γ implies that Γ is a hyperbolic group and that the orbit map $\gamma \mapsto \gamma o$ is a quasi-isometric embedding that continuously extends to a Γ-equivariant homeomorphism between the Gromov boundary $\partial\Gamma$ and limit set Λ_θ. One key feature of a Gromov hyperbolic space is that the Gromov product measures the distance between a fixed point and a geodesic, up to an additive error. The main philosophy of our proof is to establish an analogue of this property, by showing that there is a metric-like function d_ψ on Γo that is closely related to the ψ-Gromov product $\psi \circ \mathcal{G}$ on the limit set Λ_θ. For $\gamma_1, \gamma_2 \in \Gamma$, set

$$\mathsf{d}_\psi(\gamma_1 o, \gamma_2 o) = \psi(\mu(\gamma_1^{-1}\gamma_2)). \tag{1.6}$$

We prove that d_ψ satisfies the coarse triangle inequality (Theorem 4.1), using a higher rank Morse lemma due to Kapovich-Leeb-Porti [29]: there exists $D > 0$ such that for any $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$,

$$\mathsf{d}_\psi(\gamma_1 o, \gamma_3 o) \leq \mathsf{d}_\psi(\gamma_1 o, \gamma_2 o) + \mathsf{d}_\psi(\gamma_2 o, \gamma_3 o) + D. \tag{1.7}$$

This allows us to treat d_ψ as a "metric" on Γo. Moreover $(\Gamma o, \mathsf{d}_\psi)$ has a uniform thin-triangle property. That is, there exists $\delta > 0$ such that for any $\xi_1, \xi_2, \xi_3 \in \Gamma \cup \partial\Gamma$, the image of the geodesic triangle $[\xi_1, \xi_2] \cup [\xi_2, \xi_3] \cup [\xi_3, \xi_1]$ under the orbit map is δ-thin in the d_ψ-metric. On the other hand, since $(\Gamma o, \mathsf{d}_\psi)$ is not a geodesic space in general, the thin-triangle property does not imply that $(\Gamma o, \mathsf{d}_\psi)$ is a Gromov hyperbolic space. Nevertheless, investigating fine geometric properties of thin-triangles in $(\Gamma o, \mathsf{d}_\psi)$ leads us to proving that the ψ-Gromov product measures the d_ψ-distance between o and a geodesic (Proposition 6.7). That is, for $\xi \neq \eta \in \Lambda_\theta \simeq \partial\Gamma$,

$$\psi(\mathcal{G}(\xi, \eta)) = \mathsf{d}_\psi(o, [\xi, \eta]o) + O(1) \tag{1.8}$$

where $[\xi, \eta]o$ is the image of a bi-infinite geodesic $[\xi, \eta]$ in Γ connecting ξ and η under the orbit map. We also prove that shadows on the Gromov boundary $\partial\Gamma$ are comparable to shadows on Λ_θ (Proposition 7.2) and use it to establish the compatibility of the d_ψ-balls and shadows in Λ_θ (Theorem 6.2): for all large $R > 1$ there exists $c \geq 1$ such that for any $\xi \in \Lambda_\theta$ and $\gamma \in \Gamma$ on a geodesic ray in Γ toward $\xi \in \Lambda_\theta \simeq \partial\Gamma$ from the identity $e \in \Gamma$, we have

$$B_\psi(\xi, c^{-1}e^{-\psi(\mu(\gamma))}) \subset O_R(o, \gamma o) \cap \Lambda_\theta \subset B_\psi(\xi, ce^{-\psi(\mu(\gamma))}) \tag{1.9}$$

where the shadow $O_R(o, \gamma o)$ is the set of endpoints of all positive Weyl chambers based at o passing the Riemannian ball in X of radius $R > 0$ with center γo in X. Then the Ahlfors one-regularity of ν_ψ is deduced by applying the higher rank version of Sullivan's shadow lemma (Lemma 8.4). While positivity of $\mathcal{H}_\psi(\Lambda_\theta)$ is a standard consequence of the Ahlfors 1-regularity, finiteness of $\mathcal{H}_\psi(\Lambda_\theta)$ is not immediate since d_ψ is not a genuine metric. We rely on the Vitali covering type lemma for the conformal premetric d_ψ on Λ_θ (Lemma 5.6).

Organization.

- In section 2, we review some basic structures of Lie groups and θ-boundaries. The notations set up in this section will be used throughout the paper.
- In section 3, we recall the classification of Patterson-Sullivan measures of Anosov subgroups using tangent forms and some basic properties of Anosov subgroups.
- In section 4, we show that for each $\psi \in \mathfrak{a}_\theta^*$ positive on $\mathcal{L}_\theta - \{0\}$, the composition $\psi \circ \mu$ defines a metric-like function d_ψ on the Γ-orbit Γo. The coarse triangle inequality of d_ψ (Theorem 4.1) is a crucial ingredient of this paper. Its proof makes a heavy use of the notion of diamonds and the Morse lemma due to Kapovich-Leeb-Porti (Theorem 4.11).
- In section 5, we define a conformal premetric d_ψ on the limit set Λ_θ and discuss its basic properties.
- Sections 6 and 7 are devoted to the proof of the compatibility between shadows and d_ψ-balls in the limit set Λ_θ as in (1.9).
- In sections 8 and 9, we prove Theorem 1.1. In section 8, we prove that for symmetric $\psi \in \mathfrak{a}_\theta^*$, the (Γ, ψ)-Patterson-Sullivan measure is Ahlfors one-regular. In section 9, we prove that Patterson-Sullivan measures for symmetric linear forms are Hausdorff measures on the limit set, up to a constant multiple. We also prove Theorem 1.4.
- In section 10, we prove Theorem 1.7 on the estimate of the Hausdorff dimension of Λ_θ with respect to a Riemannian metric.
- In section 11, we obtain an upper bound on the growth indicator and discuss its implications on the temperedness of $L^2(\Gamma\backslash G)$.

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2. Basic structure theory of Lie groups and θ-boundaries

Throughout the paper, let G be a connected semisimple real algebraic group, more precisely, G is the identity component $\mathbf{G}(\mathbb{R})^\circ$ of the group of real points of a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. In this section, we review some basic facts about the Lie group structure of G. Let A be a maximal real split torus of G. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and set $A^+ = \exp \mathfrak{a}^+$, and a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots and Π the set of all simple roots given by the choice of $\mathfrak{a}^+$. Denote by $N_K(A)$ and $C_K(A)$ the normalizer and centralizer of A in K respectively. The Weyl group $\mathcal{W}$ is given by $N_K(A)/C_K(A)$. Consider the real vector space $\mathsf{E}^* = \mathsf{X}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathsf{X}(A)$ is the group of all real characters of A. and let E be its dual. Denote by $(,)$ a $\mathcal{W}$-invariant inner product on E. We denote by $\{\omega_\alpha : \alpha \in \Pi\}$ the (restricted) fundamental weights of Φ defined by

$$2\frac{(\omega_\alpha, \beta)}{(\beta, \beta)} = c_\alpha \delta_{\alpha,\beta} \tag{2.1}$$

where $c_\alpha = 1$ if $2\alpha \notin \Phi$ and $c_\alpha = 2$ otherwise.

Fix an element $w_0 \in N_K(A)$ of order 2 representing the longest Weyl element so that $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The map

$$\mathrm{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a} \tag{2.2}$$

is called the opposition involution. It induces an involution of Φ preserving Π, for which we use the same notation i, so that $\mathrm{i}(\alpha) = \alpha \circ \mathrm{i}$ for all $\alpha \in \Phi$.

Henceforth, we fix a non-empty subset θ of Π. Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \qquad \mathfrak{a}_\theta^+ = \mathfrak{a}_\theta \cap \mathfrak{a}^+,$$

$$A_\theta = \exp \mathfrak{a}_\theta, \quad \text{and} \quad A_\theta^+ = \exp \mathfrak{a}_\theta^+.$$

Let

$$p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$$

denote the projection invariant under all $w \in \mathcal{W}$ fixing $\mathfrak{a}_\theta$ pointwise.

Let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, $P_\theta = L_\theta N_\theta$ where L_θ is the centralizer of A_θ and N_θ is the unipotent radical of P_θ such that $\log N_\theta$ is generated by root subgroups associated to all positive roots which are not $\mathbb{Z}$-linear combinations of $\Pi - \theta$.

We set $M_\theta = K \cap P_\theta = C_K(A_\theta)$. The Levi subgroup L_θ can be written as $L_\theta = A_\theta S_\theta$ where S_θ is an almost direct product of a connected semisimple real algebraic subgroup and a compact center. Letting $B_\theta = S_\theta \cap A$

and $B_\theta^+ = \{b \in B_\theta : \alpha(\log b) \geq 0 \text{ for all } \alpha \in \Pi - \theta\}$, we have the Cartan decomposition of S_θ:

$$S_\theta = M_\theta B_\theta^+ M_\theta.$$

Note that $A = A_\theta B_\theta$ and $A^+ \subset A_\theta^+ B_\theta^+$. The space $\mathfrak{a}_\theta^* = \mathrm{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ can be identified with the subspace of $\mathfrak{a}^*$ consisting of p_θ-invariant linear forms:

$$\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi \circ p_\theta = \psi\}. \tag{2.3}$$

Hence for $\theta_1 \subset \theta_2$, we have

$$\mathfrak{a}_{\theta_1}^* \subset \mathfrak{a}_{\theta_2}^*. \tag{2.4}$$

When $\theta = \Pi$, we will omit the subscript. So $P = P_\Pi$ is a minimal parabolic subgroup and $P = MAN$.

Cartan projection. Recall the Cartan decomposition $G = KA^+K$, which means that for every $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp \mu(g) K$. The map $G \to \mathfrak{a}^+$ given by $g \mapsto \mu(g)$ is called the Cartan projection. We have

$$\mu(g^{-1}) = \mathrm{i}(\mu(g)) \quad \text{for all } g \in G. \tag{2.5}$$

Let $X = G/K$ be the associated Riemannian symmetric space, and set $o = [K] \in X$. Fix a K-invariant norm $\|\cdot\|$ on $\mathfrak{g}$ and a Riemannian metric d on X, induced from the Killing form on $\mathfrak{g}$; so that

$$d(go, ho) = \|\mu(g^{-1}h)\|$$

for any $g, h \in G$. For $p \in X$ and $R > 0$, let $B(p, R)$ denote the metric ball $\{x \in X : d(x, p) < R\}$.

Lemma 2.1. [3, Lemma 4.6] *For any compact subset $Q \subset G$, there exists a constant $C = C_Q > 0$ such that for all $g \in G$,*

$$\sup_{q_1, q_2 \in Q} \|\mu(q_1 g q_2) - \mu(g)\| < C.$$

We then write

$$\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+.$$

In view of (2.3), we have $\psi \circ \mu_\theta = \psi \circ \mu$ for all $\psi \in \mathfrak{a}_\theta^*$.

The θ-boundary $\mathcal{F}_\theta$. We set

$$\mathcal{F}_\theta = G/P_\theta \quad \text{and} \quad \mathcal{F} = G/P.$$

Let

$$\pi_\theta : \mathcal{F} \to \mathcal{F}_\theta$$

denote the canonical projection map given by $gP \mapsto gP_\theta$, $g \in G$. We set

$$\xi_\theta = [P_\theta] \in \mathcal{F}_\theta. \tag{2.6}$$

By the Iwasawa decomposition $G = KP = KAN$, the subgroup K acts transitively on $\mathcal{F}_\theta$, and hence $\mathcal{F}_\theta \simeq K/M_\theta$.

We consider the following notion of convergence of a sequence in G (or in X) to an element of $\mathcal{F}_\theta$. For a sequence $g_i \in G$, we say $g_i \to \infty$ θ-regularly if $\min_{\alpha\in\theta} \alpha(\mu(g_i)) \to \infty$ as $i \to \infty$.

Definition 2.2. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i\to\infty} g_i = \lim g_i o = \xi$ and say g_i (or $g_i o \in X$) converges to ξ if

- $g_i \to \infty$ θ-regularly; and
- $\lim_{i\to\infty} \kappa_i \xi_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_i \in K$ such that $g_i \in \kappa_i A^+ K$.

Points in general position. Let P_θ^+ be the standard parabolic subgroup of G opposite to P_θ such that $P_\theta \cap P_\theta^+ = L_\theta$. We have $P_\theta^+ = w_0 P_{\mathrm{i}(\theta)} w_0^{-1}$ and hence

$$\mathcal{F}_{\mathrm{i}(\theta)} = G/P_\theta^+.$$

For $g \in G$, we set

$$g_\theta^+ := gP_\theta \quad \text{ and } \quad g_\theta^- := gw_0 P_{\mathrm{i}(\theta)};$$

as we fix θ in the entire paper, we write $g^\pm = g_\theta^\pm$ for simplicity when there is no room for confusion. Hence for the identity $e \in G$, $(e^+, e^-) = (P_\theta, P_\theta^+) = (\xi_\theta, w_0 \xi_{\mathrm{i}(\theta)})$. The G-orbit of (e^+, e^-) is the unique open G-orbit in $G/P_\theta \times G/P_\theta^+$ under the diagonal G-action. We set

$$\mathcal{F}_\theta^{(2)} = \{(g_\theta^+, g_\theta^-) : g \in G\}. \tag{2.7}$$

Two elements $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position (or antipodal) if $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$.

Busemann maps and Gromov products. The $\mathfrak{a}$-valued Busemann map $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we define

$$\beta_\xi^\theta(g, h) := p_\theta(\beta_{\xi_0}(g, h)) \quad \text{for } \xi_0 \in \pi_\theta^{-1}(\xi); \tag{2.8}$$

this is well-defined independent of the choice of ξ_0 [49, Lemma 6.1]. We also have $\|\beta_\xi^\theta(g, h)\| \le d(go, ho)$ for all $g, h \in G$ [49, Lemma 8.9]. The Busemann map has the following properties: for all $\xi \in \mathcal{F}_\theta$ and $g_1, g_2, g_3 \in G$,

$$\text{(Invariance)} \quad \beta_\xi^\theta(g_1, g_2) = \beta_{g_3\xi}^\theta(g_3 g_1, g_3 g_2);$$

$$\text{(Cocycle property)} \quad \beta_\xi^\theta(g_1, g_2) = \beta_\xi^\theta(g_1, g_3) + \beta_\xi^\theta(g_3, g_2).$$

For $p, q \in X$ and $\xi \in \mathcal{F}_\theta$, we set $\beta_\xi^\theta(p, q) := \beta_\xi^\theta(g, h)$ where $g, h \in G$ satisfies $go = p$ and $ho = q$. It is easy to check that this is well-defined.

Definition 2.3. For $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$, we define the θ-Gromov product as

$$\mathcal{G}^\theta(\xi, \eta) = \frac{1}{2}(\beta_\xi^\theta(e, g) + \mathrm{i}(\beta_\eta^{\mathrm{i}(\theta)}(e, g)))$$

where $g \in G$ satisfies $(g^+, g^-) = (\xi, \eta)$. This does not depend on the choice of g [33, Lemma 9.11].

3. Classification of Patterson-Sullivan measures by tangent forms

Let G be a connected semisimple real algebraic group. We fix a non-empty subset θ of the set Π of all simple roots. Throughout this section, let Γ be a discrete subgroup of G. When Γ is θ-Anosov, we have a complete classification of all linear forms $\psi \in \mathfrak{a}_\theta^*$ admitting a (Γ, ψ)-Patterson-Sullivan measure ([37] [53], [33]). The goal of this section is to review this classification, in addition to recalling some basic notions such as the limit cone and the growth indicator of Γ. We refer to [33] for more details on this section.

The θ-limit set of Γ is defined as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_\theta : \ \gamma_i \in \Gamma\}$$

where $\lim \gamma_i$ is defined as in Definition 2.2. If $\Gamma < G$ is Zariski dense, then the limit set Λ_θ is the unique Γ-minimal subset of $\mathcal{F}_\theta$ ([3, Section 3.6], [49, Theorem 7.2]). Furthermore, if we set $\Lambda = \Lambda_\Pi$, then $\pi_\theta(\Lambda) = \Lambda_\theta$. For $\psi \in \mathfrak{a}_\theta^*$, a Borel probability measure ν on $\mathcal{F}_\theta$ is called a (Γ, ψ)-conformal measure if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}_\theta$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi^\theta(e,\gamma))}$$

where $\gamma_*\nu(B) = \nu(\gamma^{-1}B)$ for any Borel $B \subset \mathcal{F}_\theta$. A (Γ, ψ)-conformal measure is called a (Γ, ψ)-Patterson-Sullivan measure if it is supported on Λ_θ.

In order to discuss which linear forms ψ admits a Patterson-Sullivan measure, we need the definitions of the θ-limit cones and growth indicators.

The θ-limit cone $\mathcal{L}_\theta = \mathcal{L}_\theta(\Gamma)$ of Γ is defined as the asymptotic cone of $\mu_\theta(\Gamma)$ in $\mathfrak{a}_\theta$, that is, $u \in \mathcal{L}_\theta$ if and only if $u = \lim t_i \mu_\theta(\gamma_i)$ for some sequences $t_i \to 0$ and $\gamma_i \in \Gamma$. If Γ is Zariski dense, $\mathcal{L}_\theta$ is a convex cone with non-empty interior by [3, Section 1.2]. Recalling the convention of dropping the subscript θ when $\theta = \Pi$, we write $\mathcal{L} = \mathcal{L}_\Pi$. We then have $p_\theta(\mathcal{L}) = \mathcal{L}_\theta$.

Growth indicators. We say that Γ is θ-discrete if the restriction $\mu_\theta|_\Gamma : \Gamma \to \mathfrak{a}_\theta^+$ is proper. The θ-discreteness of Γ implies that $\mu_\theta(\Gamma)$ is a closed discrete subset of $\mathfrak{a}_\theta^+$. Indeed, Γ is θ-discrete if and only if the counting measure on $\mu_\theta(\Gamma)$ weighted with multiplicity is a Radon measure on $\mathfrak{a}_\theta^+$.

Definition 3.1 (θ-growth indicator ([48], [33]))**.** For a θ-discrete subgroup $\Gamma < G$, the θ-growth indicator $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [-\infty, \infty]$ is defined as follows: if $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi_\Gamma^\theta(u) = \|u\| \inf_{u \in \mathcal{C}} \tau_\mathcal{C}^\theta \tag{3.1}$$

where $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u, and $\psi_\Gamma^\theta(0) = 0$. Here $-\infty \le \tau_\mathcal{C}^\theta \le \infty$ denotes the abscissa of convergence of the series $\mathcal{P}_\mathcal{C}^\theta(s) = \sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$. As mentioned, we simply write $\psi_\Gamma := \psi_\Gamma^\Pi$.

This definition is independent of the choice of a norm on $\mathfrak{a}_\theta$. It was proved in ([48, Theorem 1.1.1], [33, Theorem 3.3]) that

$$\psi_\Gamma^\theta < \infty, \quad \mathcal{L}_\theta = \{\psi_\Gamma^\theta \geq 0\} \quad \text{ and } \quad \psi_\Gamma^\theta > 0 \text{ on } \operatorname{int}\mathcal{L}_\theta$$

where $\operatorname{int}\mathcal{L}_\theta$ denotes the interior of $\mathcal{L}_\theta$ in the relative topology. Moreover, ψ_Γ^θ is upper semi-continuous and concave. When $\theta = \mathrm{i}(\theta)$, it follows from (2.5) that ψ_Γ^θ is i-invariant.

We say a linear form ψ is tangent to ψ_Γ^θ (at $u \in \mathfrak{a}_\theta - \{0\}$) if $\psi \geq \psi_\Gamma^\theta$ and $\psi(u) = \psi_\Gamma^\theta(u)$. For any $u \in \operatorname{int}\mathcal{L}_\theta$, there exists $\psi \in \mathfrak{a}_\theta^*$ tangent to ψ_Γ^θ at u. Moreover, for any $\psi \in \mathfrak{a}_\theta^*$ tangent to ψ_Γ^θ at an interior direction of $\mathfrak{a}_\theta^+$, there exists a (Γ, ψ)-Patterson-Sullivan measure ([49, Theorem 8.4], [33, Proposition 5.9]).

For θ-Anosov subgroups, we have a more precise classification of Patterson-Sullivan measures in terms of tangent forms.

Definition 3.2. A finitely generated subgroup $\Gamma < G$ is θ-Anosov if there exists a constant $C > 1$ such that for all $\alpha \in \theta$ and $\gamma \in \Gamma$, we have

$$\alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C \tag{3.2}$$

where $|\cdot|$ denotes a fixed word metric on Γ.

We recall that all θ-Anosov subgroups are assumed to be non-elementary in this paper. Define

$$\mathscr{T}_\Gamma = \{\psi \in \mathfrak{a}_\theta^* : \psi \text{ is tangent to } \psi_\Gamma^\theta\}. \tag{3.3}$$

The following theorem can be deduced from [53, Theorem A], [50, Section 4] and [52, Lemma 4.8] (see [33, Theorem 12.2]):

Theorem 3.3. *Let Γ be a θ-Anosov subgroup. Then*

1. *ψ_Γ^θ is analytic, strictly concave and vertically tangent*[5]*;*
2. *For any $\psi \in \mathscr{T}_\Gamma$, there exists a unique unit vector $u = u_\psi \in \operatorname{int}\mathcal{L}_\theta$ such that $\psi(u) = \psi_\Gamma^\theta(u)$. If Γ is Zariski dense, the map $\psi \mapsto u_\psi$ is a bijection between $\mathscr{T}_\Gamma$ and $\{u \in \operatorname{int}\mathcal{L}_\theta : \|u\| = 1\}$.*

The following theorem was proved in [37, Theorem 1.3] for $\theta = \Pi$ and Γ Zariski dense. The general case follows from [53, Theorem A], [33, Theorem 1.12], and [32, Theorem 1.3, Theorem 9.4].

Theorem 3.4. *Let Γ be a θ-Anosov subgroup. For any $\psi \in \mathscr{T}_\Gamma$, there exists a unique (Γ, ψ)-Patterson-Sullivan measure on Λ_θ which we denote by $\nu_\psi = \nu_{\psi,\theta}$. The map*

$$\psi \mapsto \nu_\psi$$

is a surjection from $\mathscr{T}_\Gamma$ to the space of all Γ-Patterson-Sullivan measures. If Γ is Zariski dense, then the map $\psi \mapsto \nu_\psi$ is bijective. Moreover, if $\psi_1 \neq \psi_2$ in $\mathscr{T}_\Gamma$, then ν_{ψ_1} and ν_{ψ_2} are mutually singular to each other.

[5]It means that there is no linear form tangent to ψ_Γ^θ at some $u \in \partial\mathcal{L}_\theta$.

Remark 3.5. One immediate consequence of the last statement of Theorem 3.4 is that at most one Patterson-Sullivan measure can be a Hausdorff measure on Λ_θ with respect to a fixed metric (e.g., Riemannian metric).

When $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta - \{0\}$, the abscissa of convergence of the ψ-Poincaré series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\psi(\mu(\gamma))}$$

is a well-defined positive number we denote it by δ_ψ [33, Lemma 4.3]. Equivalently, δ_ψ is also given by (1.3).

Lemma 3.6. [33, Lemma 4.5] *If $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta - \{0\}$, then*

$$\delta_\psi \psi \in \mathscr{T}_\Gamma.$$

In particular, $\psi \in \mathscr{T}_\Gamma$ if and only if $\delta_\psi = 1$.

Since $\mu(g^{-1}) = \mathrm{i}(\mu(g))$ for all $g \in G$, we have that Γ is θ-Anosov if and only if Γ is $\theta \cup \mathrm{i}(\theta)$-Anosov. If Γ is θ-Anosov, then the canonical projection map $p : \Lambda_{\theta\cup\mathrm{i}(\theta)} \to \Lambda_\theta$ is a Γ-equivariant homeomorphism. Recalling that $\mathfrak{a}_\theta^*$ can be considered as a subset of $\mathfrak{a}_{\theta\cup\mathrm{i}(\theta)}^*$ from (2.4), we recall the following which will be of use.

Lemma 3.7. [33, Lemma 9.5] *Let Γ be a θ-Anosov subgroup. For any $\psi \in \mathscr{T}_\Gamma$, the measure $\nu_{\psi,\theta}$ coincides with the push-forward of $\nu_{\psi,\theta\cup\mathrm{i}(\theta)}$ by p.*

Gromov hyperbolic space and quasi-isometry. We collect a few basic facts about θ-Anosov subgroups which will be used repeatedly.

Recall that a geodesic metric space[6] (Z, d_Z) is called a Gromov hyperbolic space if it satisfies a uniformly thin-triangle property, that is, there exists $T > 0$ such that for any geodesic triangle in Z, one side of the triangle is contained in the T-neighborhood of the union of two other sides. We denote by ∂Z the Gromov boundary of Z, which is the equivalence classes of geodesic rays. For any $z_1 \neq z_2 \in Z \cup \partial Z$, there may be more than one geodesic connecting z_1 and z_2. By the notation $[z_1, z_2]$, we mean "a" geodesic in Z connecting z_1 to z_2. For $w \in Z$, the nearest-point projection of w to a geodesic $[z_1, z_2]$ is any point $w' \in [z_1, z_2]$ satisfying $d_Z(w, w') = \inf\{d_Z(w, z) : z \in [z_1, z_2]\}$. This is coarsely well-defined. One can refer to [11] for basics on Gromov hyperbolic spaces. Recall that d denotes the Riemannian metric on $X = G/K$.

Theorem 3.8 ([29, Corollary 1.6], [28, Proposition 5.16, Lemma 5.23], see also [24])**.** *Let Γ be a θ-Anosov subgroup. Fix a word metric d_Γ on Γ with respect to a finite symmetric generating set. We have:*

1. *$(\Gamma, \mathsf{d}_\Gamma)$ is a Gromov hyperbolic space*[7]*;*
2. *$\mathcal{L}_\theta - \{0\} \subset \operatorname{int} \mathfrak{a}_\theta^+$;*

---

[6] that is, there exists a geodesic between any two distinct points.

[7] note that the metric space $(\Gamma, \mathsf{d}_\Gamma)$ is clearly a proper geodesic space

(3) *The orbit map* $(\Gamma, \mathsf{d}_\Gamma) \to (\Gamma o, d)$ *given by* $\gamma \mapsto \gamma o$ *is a quasi-isometry, i.e., there exist* $Q = Q_\Gamma \geq 1$ *such that for all* $\gamma_1, \gamma_2 \in \Gamma$,

$$Q^{-1} \cdot \mathsf{d}_\Gamma(\gamma_1, \gamma_2) - Q \leq d(\gamma_1 o, \gamma_2 o) \leq Q \cdot \mathsf{d}_\Gamma(\gamma_1, \gamma_2) + Q.$$

(4) *The orbit map* $\Gamma \to \Gamma o$ *extends to the* Γ*-equivariant homeomorphism* $f : \Gamma \cup \partial\Gamma \to \Gamma o \cup \Lambda_\theta$ *where* $\Gamma \cup \partial\Gamma$ *is given the topology of the Gromov compactification, so that two distinct points in* $\partial\Gamma$ *are mapped to points in general position.*

We will henceforth identity $\partial\Gamma$ and Λ_θ using f. For any $\xi \neq \eta \in \Gamma \cup \partial\Gamma$, note that $f([\xi, \eta]) = [\xi, \eta]o$ is the image of $[\xi, \eta]$ under the orbit map.

4. Metric-like functions on Γ-orbits and diamonds

We fix a non-empty subset $\theta \subset \Pi$. In this section, we assume that θ is symmetric, i.e., $\theta = \mathrm{i}(\theta)$. Recall the notation $X = G/K$ and $o = [K] \in X$.

For a linear form $\psi \in \mathfrak{a}_\theta^*$, define $\mathsf{d}_\psi : X \times X \to \mathbb{R}$ as follows: for $g, h \in G$,

$$\mathsf{d}_\psi(go, ho) := \psi(\mu(g^{-1}h)) = \psi(\mu_\theta(g^{-1}h)). \tag{4.1}$$

Since the Cartan projection μ is bi-K-invariant, d_ψ is a well-defined left G-invariant function.

The main goal of this section is to prove the following theorem saying that when Γ is θ-Anosov, d_ψ behaves like a metric, restricted to the Γ-orbit Γo for a proper class of ψ's:

Theorem 4.1 (Coarse triangle inequality)**.** *Let* Γ *be a* θ*-Anosov subgroup. Let* $\psi \in \mathfrak{a}_\theta^*$ *be such that* $\psi > 0$ *on* $\mathcal{L}_\theta - \{0\}$. *Then there exists a constant* $D = D_\psi > 0$ *such that for all* $\gamma_1, \gamma_2, \gamma \in \Gamma$,

$$\mathsf{d}_\psi(\gamma_1 o, \gamma_2 o) \leq \mathsf{d}_\psi(\gamma_1 o, \gamma o) + \mathsf{d}_\psi(\gamma o, \gamma_2 o) + D.$$

Indeed, we prove Theorem 4.1 in a greater generality where the orbit Γo is replaced by the image of a *uniformly regular* quasi-isometric embedding of a geodesic metric space into X.

Coarse triangle inequalities for uniformly regular quasi-isometric embeddings. We set $\mathcal{W}_\theta = \{w \in \mathcal{W} : w(\theta) = \theta\}$. We define a closed cone $\mathcal{C}$ in $\mathfrak{a}^+$ to be θ-admissible if the following three conditions hold:

(1) $\mathcal{C}$ is i-invariant: $\mathrm{i}(\mathcal{C}) = \mathcal{C}$;
(2) $\mathcal{W}_\theta \cdot \mathcal{C} = \bigcup_{w \in \mathcal{W}_\theta} \mathrm{Ad}_w \, \mathcal{C}$ is convex;
(3) $\mathcal{C} \cap \left(\bigcup_{\alpha \in \theta} \ker \alpha\right) = \{0\}$.

For a θ-admissible cone $\mathcal{C}$, we say that an ordered pair (x_1, x_2) of distinct points in X is *$\mathcal{C}$-regular* if for $g_1, g_2 \in G$ such that $g_1 o = x_1$ and $g_2 o = x_2$, we have

$$\mu(g_1^{-1} g_2) \in \mathcal{C}.$$

In this case, $x_2 = g_2 o \in g_1 K(\exp \mathcal{C})o$ and hence for some $g \in g_1 K$, $x_1 = g_1 o = go$ and $x_2 \in g(\exp \mathcal{C})o$. Note that if (x_1, x_2) is $\mathcal{C}$-regular, then (x_2, x_1) is $\mathrm{i}(\mathcal{C})$-regular and hence $\mathcal{C}$-regular by the i-invariance of $\mathcal{C}$.

Definition 4.2. Let (Z, d_Z) be a metric space and $f : Z \to X$ be a map. For a cone $\mathcal{C} \subset \mathfrak{a}^+$ and a constant $B \geq 0$, f is called $(\mathcal{C}, B)$*-regular* if the pair $(f(z_1), f(z_2))$ is $\mathcal{C}$-regular for all $z_1, z_2 \in Z$ with $d_Z(z_1, z_2) \geq B$. We simply say f is $\mathcal{C}$-regular if it is $(\mathcal{C}, B)$-regular for some $B \geq 0$.

Theorem 4.1 will be deduced as a special case of the following theorem: we write $\mathcal{C}_\theta = p_\theta(\mathcal{C})$.

Theorem 4.3. *Let Z be a geodesic metric space and $\mathcal{C} \subset \mathfrak{a}^+$ a θ-admissible cone. Let $f : Z \to X$ be a $\mathcal{C}$-regular quasi-isometric embedding*[8]*. If $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{C}_\theta - \{0\}$, then there exists a constant $D = D_\psi \geq 0$ such that for all $x_1, x_2, x_3 \in f(Z)$,*

$$\mathsf{d}_\psi(x_1, x_3) \leq \mathsf{d}_\psi(x_1, x_2) + \mathsf{d}_\psi(x_2, x_3) + D.$$

Proof of Theorem 4.1 assuming Theorem 4.3. Since $\theta = \mathrm{i}(\theta)$ by hypothesis, it follows from (2.5) that $\mathrm{i}\,|_{\mathfrak{a}_\theta}$ is an involution preserving $\mathcal{L}_\theta$. Since ψ is positive on $\mathcal{L}_\theta - \{0\}$ and $\mathcal{L}_\theta - \{0\} \subset \operatorname{int} \mathfrak{a}_\theta^+$ (Theorem 3.8(2)), we can choose a slightly larger closed cone $\mathcal{C}_0 \subset \operatorname{int} \mathfrak{a}_\theta^+ \cup \{0\}$ satisfying

1. $\mathcal{L}_\theta - \{0\} \subset \operatorname{int} \mathcal{C}_0$;
2. $\mathrm{i}(\mathcal{C}_0) = \mathcal{C}_0$;
3. $\psi > 0$ on $\mathcal{C}_0 - \{0\}$.

Define

$$\mathcal{C} \coloneqq p_\theta^{-1}(\mathcal{C}_0) \cap \mathfrak{a}^+.$$

Using the fact that $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ is $\mathcal{W}_\theta$-equivariant, we have that $\mathcal{W}_\theta \cdot \mathcal{C} = \mathcal{W}_\theta \cdot (p_\theta^{-1}(\mathcal{C}_0) \cap \mathfrak{a}^+) = p_\theta^{-1}(\mathcal{C}_0) \cap (\mathcal{W}_\theta \cdot \mathfrak{a}^+)$. Since both $\mathcal{W}_\theta \cdot \mathfrak{a}^+$ and $p_\theta^{-1}(\mathcal{C}_0)$ are convex cones, it follows that their intersection $\mathcal{W}_\theta \cdot \mathcal{C}$ is also convex. After removing some small enough conical neighborhoods of $\bigcup_{\alpha \in \theta} \ker \alpha$ from $\mathcal{C}$, we can assume that

$$\mathcal{C} \cap \bigcup_{\alpha \in \theta} \ker \alpha = \{0\}$$

while keeping the convexity of $\mathcal{W}_\theta \cdot \mathcal{C}$. Moreover, we can choose those neighborhoods thin so that $\mathcal{C}$ still contains a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$. The resulting $\mathcal{C}$ is therefore θ-admissible and $\psi > 0$ on $\mathcal{C} - \{0\}$. Since the orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (X, d)$, $\gamma \mapsto \gamma o$, is a quasi-isometric embedding as stated in Theorem 3.8(3), Theorem 4.1 follows from Theorem 4.3 once we prove that the orbit map is a $\mathcal{C}$-regular embedding, as below. $\square$

Lemma 4.4. *Let $\mathcal{C}$ be a closed cone containing a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$. Then the orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (X, d)$ is $\mathcal{C}$-regular.*

Proof. Suppose not. Then there exist two sequences $\{\gamma_i\}, \{\gamma_i'\} \subset \Gamma$ such that $\mathsf{d}_\Gamma(\gamma_i, \gamma_i') = |\gamma_i^{-1}\gamma_i'| > i$ and $\mu(\gamma_i^{-1}\gamma_i') \notin \mathcal{C}$ for all $i \geq 1$. Setting

[8]A map $f : (Z, d_Z) \to (Y, d_Y)$ between metric spaces is called a Q-quasi-isometric embedding for $Q \geq 1$ if for all $z_1, z_2 \in Z$, $Q^{-1} d_Z(z_1, z_2) - Q \leq d_Y(f(z_1), f(z_2)) \leq Q d_Z(z_1, z_2) + Q$. A map is called a quasi-isometric embedding if it is a Q-quasi-isometric embedding for some $Q \geq 1$.

$g_i = \gamma_i^{-1}\gamma_i' \in \Gamma$, we then have that $\frac{\mu(g_i)}{\|\mu(g_i)\|} \notin \mathcal{C}$ for all $i \geq 1$. Since a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$ is contained in $\mathcal{C}$, no limit of the sequence $\frac{\mu(g_i)}{\|\mu(g_i)\|}$ is contained in $\mathcal{L}$. On the other hand, since $|g_i| \to \infty$, we have $\|\mu(g_i)\| \to \infty$ and hence any limit of the sequence $\frac{\mu(g_i)}{\|\mu(g_i)\|}$ must belong to the asymptotic cone of $\mu(\Gamma)$, that is, $\mathcal{L}$. This yields a contradiction. $\square$

The rest of this section is devoted to the proof of Theorem 4.3. We begin by recalling the following theorem; in particular, the metric space Z in Theorem 4.3 is always Gromov hyperbolic.

Theorem 4.5. [29, Theorem 1.4] *Let Z and $f : Z \to X$ be as in Theorem 4.3. Then Z is Gromov hyperbolic. If Z is proper in addition, then f continuously extends to*

$$f : \bar{Z} \to X \cup \mathcal{F}_\theta$$

where $\bar{Z} = Z \cup \partial Z$ is the Gromov compactification and f maps two distinct points in ∂Z to points in general position.

Diamonds. The notion of diamonds in X, due to Kapovich-Leeb-Porti, plays a key role in the proof of Theorem 4.3. We fix

$$\text{a } \theta\text{-admissible cone } \mathcal{C} \subset \mathfrak{a}^+$$

in the following. For a $\mathcal{C}$-regular pair (x_1, x_2) of points in X, define the $\mathcal{C}$*-cone* with the tip at x_1 containing x_2 to be

$$V_\mathcal{C}(x_1, x_2) = gM_\theta(\exp \mathcal{C})o,$$

where $g = g(x_1, x_2) \in G$ is any element such that $x_1 = go$ and $x_2 \in g(\exp \mathcal{C})o$; it is easy to check such g always exists and this definition is independent of the choice of g. For any $h \in G$, we have $hV_\mathcal{C}(x_1, x_2) = V_\mathcal{C}(hx_1, hx_2)$.

Definition 4.6 (Diamonds)**.** For a $\mathcal{C}$-regular pair (x_1, x_2) of points in X, the $\mathcal{C}$*-diamond* with tips at x_1 and x_2 is defined as

$$\Diamond_\mathcal{C}(x_1, x_2) = V_\mathcal{C}(x_1, x_2) \cap V_\mathcal{C}(x_2, x_1).$$

The $\mathcal{C}$-cones and $\mathcal{C}$-diamonds are convex subsets of X, see [28, Propositions 2.10 and 2.13]. Note also the equivariance property that for $h \in G$, $h\Diamond_\mathcal{C}(x_1, x_2) = \Diamond_\mathcal{C}(hx_1, hx_2)$. It follows that for any $\mathcal{C}$-regular pair (x_1, x_2), the diamond $\Diamond_\mathcal{C}(x_1, x_2)$ is of the form $h\Diamond_\mathcal{C}(o, ao)$ for some $a \in \exp \mathcal{C}$ and $h \in G$. Therefore the following example describes all diamonds up to translations.

Example 4.7. For $a \in \exp \mathcal{C}$, the diamond $\Diamond_\mathcal{C}(o, ao)$ can be explicitly described as follows. First note that as we can take $g(o, ao) = e$, we have $V_\mathcal{C}(o, ao) = M_\theta(\exp \mathcal{C})o$. Recalling that $\mathrm{i} = -\operatorname{Ad}_{w_0}$, we also have $ao = aw_0o$ and $o = (aw_0)(w_0^{-1}a^{-1}w_0)o \in aw_0(\exp \mathcal{C})o$. So we can take

$g(ao, o) = aw_0$. Since $w_0 M_\theta w_0^{-1} = M_\theta$ and $w_0(\exp \mathcal{C}) w_0^{-1} = \exp(-\mathcal{C})$, we have $V_{\mathcal{C}}(ao, o) = aw_0 M_\theta (\exp \mathcal{C}) o = aM_\theta \exp(-\mathcal{C}) o$. Therefore

$$\Diamond_{\mathcal{C}}(o, ao) = M_\theta(\exp \mathcal{C}) o \cap aM_\theta \exp(-\mathcal{C}) o.$$

See Figure 1.

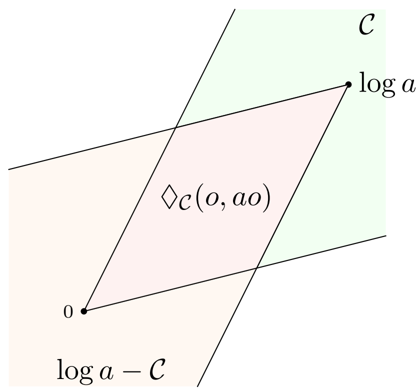

FIGURE 1. Diamond drawn in $\mathfrak{a}$

Lemma 4.8 (Simultaneously nesting property)**.** *If* (x_1, x_2) *is* $\mathcal{C}$*-regular, then for any* $x \in \Diamond_{\mathcal{C}}(x_1, x_2)$*, there exist* $g \in G$ *and* $a \in \exp \mathcal{C}$ *such that*

$$x_1 = go, \quad x = gao, \quad x_2 \in gM_\theta(\exp \mathcal{C}) o \cap ga(M_\theta \exp \mathcal{C}) o.$$

Proof. We may first assume that $x_1 = o$. By the $\mathcal{C}$-regularity of the pair (x_1, x_2), we have $x_2 \in K(\exp \mathcal{C}) o$. By multiplying an element of K to x_1 and x_2, we may also assume that $x_1 = o$ and $x_2 \in (\exp \mathcal{C}) o$, and hence $x \in M_\theta(\exp \mathcal{C}) o$. We again multiply an element of M_θ to x_1, x_2 and x if necessary so that we have $x_1 = o$, $x_2 \in M_\theta(\exp \mathcal{C}) o$, and $x = ao$ for some $a \in \exp \mathcal{C}$. Then it suffices to show that $x_2 = aka'o$ for some $k \in M_\theta$ and $a' \in \exp \mathcal{C}$. We write $x_2 = ma_0 o$ for $m \in M_\theta$ and $a_0 \in \exp \mathcal{C}$. We then have $o \in V_{\mathcal{C}}(x_2, o) = ma_0 k_0 M_\theta(\exp \mathcal{C}) o$ for some $k_0 \in K$. Hence we have

$$k_0^{-1} w_0^{-1}(w_0 a_0^{-1} w_0^{-1}) \in M_\theta(\exp \mathcal{C}) K.$$

This implies $k_0^{-1} \in M_\theta w_0$ and hence $k_0 \in w_0 M_\theta$. Since $ao \in V_{\mathcal{C}}(x_2, o)$ as well, we now have $ao \in ma_0 w_0 M_\theta(\exp \mathcal{C}) o$. Then for some $k \in K$, we have

$$ak \in ma_0 w_0 M_\theta \exp \mathcal{C} w_0^{-1} = ma_0 M_\theta \exp(-\mathcal{C}).$$

Hence for some $a' \in \exp \mathcal{C}$, we have

$$aka' \in ma_0 M_\theta.$$

Looking at G/P_θ, we have $kP_\theta = a^{-1} ma_0 M_\theta a'^{-1} P_\theta = P_\theta$. Therefore $k \in M_\theta$. Since $x_2 = ma_0 o = aka'o$, the claim follows. $\square$

Lemma 4.9. *For any $\mathcal{C}$-regular pair (g_1o, g_2o) with $g_1, g_2 \in G$ and for any $go \in \lozenge_{\mathcal{C}}(g_1o, g_2o)$ with $g \in G$,*

$$\mu_\theta(g_1^{-1}g) + \mu_\theta(g^{-1}g_2) = \mu_\theta(g_1^{-1}g_2). \tag{4.2}$$

Proof. By Lemma 4.8, there exists $h \in G$, $a, \tilde{a}, a' \in \exp \mathcal{C}$ and $\tilde{k}, k \in M_\theta$ such that $g_1o = ho$, $go = hao$, and $g_2o = h\tilde{k}\tilde{a}o = haka'o$. Without loss of generality, we may assume $h = e$ in proving (4.2).

We write

$$a = a_1a_2 \in A_\theta^+ B_\theta^+ \quad \text{and} \quad a' = a_1'a_2' \in A_\theta^+ B_\theta^+.$$

We then have

$$aka' = a_2ka_2'(a_1a_1').$$

Since $a_2ka_2' \in S_\theta$, we can write its Cartan decomposition $a_2ka_2' = mbm' \in M_\theta B_\theta^+ M_\theta$, and hence

$$aka' = m(ba_1a_1')m'.$$

Let $w \in \mathcal{W}$ be a Weyl element such that $ba_1a_1' \in wA^+w^{-1}$. Since $\tilde{a} = \exp \mu(aka')$, we must have $ba_1a_1' = w\tilde{a}w^{-1}$. Hence we have

$$g_2o = aka'o = mw\tilde{a}o.$$

On the other hand, we also have $g_2o = \tilde{k}\tilde{a}o$ where $\tilde{k} \in M_\theta$. This implies $mw \in M_\theta$; in particular, $w \in M_\theta$. Therefore $\tilde{a} = w^{-1}ba_1a_1'w = (w^{-1}bw)(a_1a_1') \in B_\theta A_\theta^+$, which implies

$$p_\theta(\log \tilde{a}) = \log a_1 + \log a_1' = p_\theta(\log a) + p_\theta(\log a'). \tag{4.3}$$

Since

$$\mu_\theta(g_1^{-1}g) = p_\theta(\log a), \quad \mu_\theta(g^{-1}g_2) = p_\theta(\log a'), \text{ and } \quad \mu_\theta(g_1^{-1}g_2) = p_\theta(\log \tilde{a}),$$

this finishes the proof. □

As an immediate corollary, we get that d_ψ is additive on each diamond for any $\psi \in \mathfrak{a}_\theta^*$:

Lemma 4.10 (Additivity of d_ψ on diamonds)**.** *Let $\psi \in \mathfrak{a}_\theta^*$. For any $\mathcal{C}$-regular pair (x_1, x_2) and for any $x \in \lozenge_{\mathcal{C}}(x_1, x_2)$, we have*

$$\mathsf{d}_\psi(x_1, x) + \mathsf{d}_\psi(x, x_2) = \mathsf{d}_\psi(x_1, x_2).$$

KLP Morse lemma. The Morse lemma due to Kapovich-Leeb-Porti, which we will call the KLP Morse lemma, is stated as follows [29, Theorem 5.16, Corollary 5.28]: the image of an interval in $\mathbb{R}$ under a Q-quasi-isometry is called a Q-quasi-geodesic.

Theorem 4.11 (KLP Morse lemma)**.** *Let $\mathcal{C}, \mathcal{C}' \subset \mathfrak{a}^+$ be θ-admissible cones such that $\mathcal{C}'$ contains a neighborhood of $\mathcal{C} - \{0\}$. Let $Q, B \geq 1$ be constants. There exists a constant $D_0 = D_0(\mathcal{C}, \mathcal{C}', Q, B) \geq 0$ so that the following holds: let $I \subset \mathbb{R}$ be an interval and $c : I \to X$ a $(\mathcal{C}, B)$-regular Q-quasi-geodesic.*

1. *If $I = [a, b]$ with $b - a \geq B$, then the image $c(I)$ is contained in the D_0-neighborhood of the diamond $\lozenge_{\mathcal{C}'}(c(a), c(b))$.*

(2) *If* $I = [a, \infty)$ *for some* $a \in \mathbb{R}$*, then* $c(I)$ *is contained in the* D_0*-neighborhood of the cone* $gM_\theta(\exp \mathcal{C}')o$ *where* $g \in G$ *is such that* $go = c(a)$ *and* $g^+ = c(\infty) \in \mathcal{F}_\theta$*.*

(3) *If* $I = \mathbb{R}$*, then* $c(I)$ *is contained in the* D_0*-neighborhood of the parallel set* $gM_\theta Ao$ *where* $g \in G$ *is such that* $g^\pm = c(\pm\infty) \in \mathcal{F}_\theta$*.*

We note that the above applies for an interval in $\mathbb{Z}$*, as any* $\mathcal{C}$*-regular quasi-isometric embedding* $c : I \cap \mathbb{Z} \to X$ *can be extended to a* $\mathcal{C}$*-regular quasi-geodesic* $I \to X$ *simply by setting* $c(t) := c(\lfloor t \rfloor)$ *where* $\lfloor t \rfloor$ *is the largest integer not bigger than* t*.*

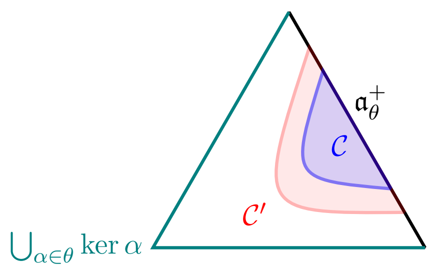

FIGURE 2. Choice of $\mathcal{C}'$ viewed on the unit sphere of $\mathfrak{a}^+$

As an application of Theorem 4.11, we get the following:

Corollary 4.12. *Given* $\mathcal{C}, \mathcal{C}', Q, B \geq 1$ *as in Theorem 4.11 and* $\psi \in \mathfrak{a}_\theta^*$*, there exists a constant* $D_1 = D_1(\mathcal{C}, \mathcal{C}', Q, B, \psi) \geq 0$ *so that the following holds: let* $I \subset \mathbb{R}$ *be an interval and* $c : I \to X$ *a* $(\mathcal{C}, B)$*-regular* Q*-quasi-geodesic. Then for all* $a \leq t \leq b$ *in* I*, we have*

$$|\mathsf{d}_\psi(c(a), c(b)) - \mathsf{d}_\psi(c(a), c(t)) - \mathsf{d}_\psi(c(t), c(b))| \leq D_1. \tag{4.4}$$

Proof. Suppose that $b - a \geq B$. Since c is $(\mathcal{C}, B)$-regular, the pair $(c(a), c(b))$ is $\mathcal{C}$-regular. Applying Theorem 4.11, we obtain that the image of $c : [a, b] \to X$ lies in the D_0-neighborhood of $\Diamond_{\mathcal{C}'}(c(a), c(b))$. For each $a < t < b$, choose $x_t \in \Diamond_{\mathcal{C}'}(c(a), c(b))$ so that $d(x_t, c(t)) \leq D_0$. Hence by Lemma 4.10,

$$\mathsf{d}_\psi(c(a), x_t) + \mathsf{d}_\psi(x_t, c(b)) = \mathsf{d}_\psi(c(a), c(b)). \tag{4.5}$$

For each $a \leq t \leq b$, write $c(t) = g_t o$ and $x_t = h_t o$ for $g_t, h_t \in G$. We then have $\|\mu(h_t^{-1} g_t)\| \leq D_0$. By applying Lemma 2.1 to a compact subset $\{g \in G : \|\mu(g)\| \leq D_0\}$, we have for all $a < t < b$,

$$|\mathsf{d}_\psi(c(a), x_t) - \mathsf{d}_\psi(c(a), c(t))| = |\psi(\mu(g_a^{-1} h_t) - \mu(g_a^{-1} g_t))| \leq C$$

where $C > 0$ is a uniform constant depending only on ψ and D_0. Similarly, we have $|\mathsf{d}_\psi(x_t, c(b)) - \mathsf{d}_\psi(c(t), c(b))| \leq C$. By (4.5), this implies that

$$|\mathsf{d}_\psi(c(a), c(b)) - \mathsf{d}_\psi(c(a), c(t)) - \mathsf{d}_\psi(c(t), c(b))| \leq 2C.$$

Setting $D_1 = 2C + 3\|\psi\|(QB + Q)$ where $\|\psi\|$ is the operator norm of ψ, we have shown that (4.4) holds whenever $b - a \geq B$. If $b - a < B$, then the image of $c([a, b])$ has diameter smaller than $Q(b - a) + Q < QB + Q$. Then

$$\mathsf{d}_\psi(c(t_1), c(t_2)) < \|\psi\|(QB + Q)$$

for all $t_1, t_2 \in [a, b]$, and hence the left hand side of (4.4) is bounded above by $3\|\psi\|(QB + Q) \leq D_1$. This completes the proof. $\square$

We are ready to give:

Proof of Theorem 4.3. Let $f : Z \to X$ be as in Theorem 4.3. Let $\psi \in \mathfrak{a}_\theta^*$ be such that $\psi > 0$ on $\mathcal{C} - \{0\}$. Choose a θ-admissible cone $\mathcal{C}' \subset \mathfrak{a}^+$ which contains a neighborhood of $\mathcal{C} - \{0\}$ in $\mathfrak{a}^+$ and such that $\psi > 0$ on $\mathcal{C}' - \{0\}$. Let $x_1, x_2, x_3 \in f(Z)$ be a triple of distinct points. We choose $z_1, z_2, z_3 \in Z$ such that $x_i = f(z_i)$ for $i = 1, 2, 3$. Choose geodesics c_1 and c_2 in Z connecting z_1 to z_2 and z_2 to z_3 respectively. By Theorem 4.5, (Z, d_Z) is Gromov hyperbolic. We denote by z the nearest-point projection of z_2 to a geodesic segment connecting z_1 and z_3. Then by the Gromov hyperbolicity of (Z, d_Z), there exists a uniform constant $\delta > 0$ so that the δ-neighborhood of z intersects both geodesics c_1 and c_2. We choose two points $y_1 \in c_1$ and $y_2 \in c_2$ which are δ-close to z. We concatenate the segment of c_1 connecting z_1 and y_1, a geodesic connecting y_1 and y_2, and the segment of c_2 connecting y_2 and z_3, and denote the concatenated path by c. We can parameterize $c : [0, b] \to Z$ so that c is a q-quasi-geodesic for some $b > 0$ and uniform $q \geq 1$ by the Gromov hyperbolicity of (Z, d_Z) and the choice of y_1 and y_2.

Since f is a $\mathcal{C}$-regular quasi-isometric embedding, so is $f \circ c$. Hence we get

$$\mathsf{d}_\psi(x_1, x_3) \leq \mathsf{d}_\psi(x_1, f(y_1)) + \mathsf{d}_\psi(f(y_1), x_3) + D_1 \tag{4.6}$$

where D_1 is the constant given by Corollary 4.12. Applying Corollary 4.12 to the restriction of $f \circ c$ to the interval $[c^{-1}(y_1), b]$ again, we have

$$\mathsf{d}_\psi(f(y_1), x_3) \leq \mathsf{d}_\psi(f(y_1), f(y_2)) + \mathsf{d}_\psi(f(y_2), x_3) + D_1. \tag{4.7}$$

Since $d_Z(y_1, y_2) \leq 2\delta$, combining (4.6) and (4.7) yields

$$\mathsf{d}_\psi(x_1, x_3) \leq \mathsf{d}_\psi(x_1, f(y_1)) + \mathsf{d}_\psi(f(y_2), x_3) + D_1'. \tag{4.8}$$

where $D_1' := \sup\{\mathsf{d}_\psi(f(w_1), f(w_2)) : d_Z(w_1, w_2) \leq 2\delta\} + 2D_1 < \infty$.

Since f is $\mathcal{C}$-regular and $\psi > 0$ on $\mathcal{C} - \{0\}$, there exists $D_2 > 0$ such that

$$\mathsf{d}_\psi(f(w_1), f(w_2)) \geq -D_2 \tag{4.9}$$

for all $w_1, w_2 \in Z$; indeed, if f is $(\mathcal{C}, B)$-regular for some $B \geq 0$, then $\mathsf{d}_\psi(f(w_1), f(w_2)) \geq 0$ whenever $d_Z(w_1, w_2) \geq B$, and $\sup\{|\mathsf{d}_\psi(f(w_1), f(w_2))| : d_Z(w_1, w_2) < B\}$ is bounded by a uniform constant depending only on B, the quasi-isometry constant of f, and $\|\psi\|$.

Hence applying (4.9) and Corollary 4.12 to $f(c_1)$, we have

$$\begin{aligned} \mathsf{d}_\psi(x_1, f(y_1)) &\leq \mathsf{d}_\psi(x_1, f(y_1)) + \mathsf{d}_\psi(f(y_1), x_2) + D_2 \\ &\leq \mathsf{d}_\psi(x_1, x_2) + D_1 + D_2. \end{aligned} \tag{4.10}$$

Similarly, we also get

$$\mathsf{d}_\psi(f(y_2), x_3) \le \mathsf{d}_\psi(x_2, x_3) + D_1 + D_2. \tag{4.11}$$

Combining (4.8), (4.10) and (4.11), we obtain

$$\mathsf{d}_\psi(x_1, x_3) \le \mathsf{d}_\psi(x_1, x_2) + \mathsf{d}_\psi(x_2, x_3) + D_1' + 2(D_1 + D_2).$$

This completes the proof of Theorem 4.3. □

We state the following consequence of the KLP Morse lemma applied to Anosov subgroups:

Theorem 4.13 (Morse lemma for Anosov subgroups)**.** *Let Γ be a θ-Anosov subgroup and $f : \Gamma \cup \partial\Gamma \to \Gamma o \cup \Lambda_\theta$ be the extension of the orbit map $\gamma \mapsto \gamma o$ given in Theorem 3.8(4). Then there exist a cone $\mathcal{C} \subset \mathfrak{a}^+$ and constants $B, D_0 \ge 1$ such that for any geodesic $[\xi, \eta]$ in Γ, the following holds:*

1. *If $\xi, \eta \in \Gamma$ and $\mathsf{d}_\Gamma(\xi, \eta) \ge B$, then $f([\xi, \eta])$ is contained in the D_0-neighborhood of the diamond $\lozenge_{\mathcal{C}}(f(\xi), f(\eta))$.*
2. *If $\xi \in \Gamma$ and $\eta \in \partial\Gamma$, then $f([\xi, \eta])$ is contained in the D_0-neighborhood of $gM_\theta(\exp \mathcal{C})o$ where $g \in G$ is such that $go = \xi$ and $gP_\theta = f(\eta)$.*
3. *If $\xi, \eta \in \partial\Gamma$, then $f([\xi, \eta])$ is contained in the D_0-neighborhood of $gM_\theta A o$ where $g \in G$ is such that $gP_\theta = f(\xi)$ and $gw_0P_\theta = f(\eta)$.*

Moreover, the cone $\mathcal{C}$ can be taken arbitrarily close to $\mathcal{L}$ as long as it contains a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$.

Proof. Let $\mathcal{C} \subset \mathfrak{a}^+$ be the θ-admissible cone as in the proof of Theorem 4.1, and choose a θ-admissible cone $\mathcal{C}' \subset \mathfrak{a}^+$ which contains a neighborhood of $\mathcal{C} - \{0\}$ in $\mathfrak{a}^+$. Then by Lemma 4.4 and Theorem 3.8(3), the orbit map $f|_\Gamma$ is a $(\mathcal{C}, B)$-regular Q-quasi-isometry between $(\Gamma, \mathsf{d}_\Gamma)$ and $(\Gamma o, d)$ for some $B, Q \ge 1$. Let $D_0 = D_0(\mathcal{C}, \mathcal{C}', Q, B)$ be as given by Theorem 4.11.

Now note that any geodesic $[\xi, \eta]$ in $(\Gamma, \mathsf{d}_\Gamma)$ can be written as $[\xi, \eta] = \{\gamma_i : i \in I\}$ for an interval I in $\mathbb{Z}$, and $\iota : i \mapsto \gamma_i$ is an isometry between I and $[\xi, \eta]$. Since $c := f \circ \iota$ is a $(\mathcal{C}, B)$-regular Q-quasi-geodesic, we can apply Theorem 4.11 which implies the above claims (1)-(3) where the cone $\mathcal{C}$ in the statement is given by $\mathcal{C}'$ in this proof. Note from the proof of Theorem 4.1 that the cone $\mathcal{C}'$ can be taken arbitrarily close to the limit cone $\mathcal{L}$ of Γ as long as $\mathcal{C}'$ contains a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$. □

5. Conformal premetrics on limit sets

Let Γ be a θ-Anosov subgroup of a connected semisimple real algebraic group G. We assume $\theta = \mathrm{i}(\theta)$ in this section. Fix a linear form $\psi \in \mathfrak{a}_\theta^*$ positive on $\mathcal{L}_\theta - \{0\}$. The goal of this section is to define a premetric d_ψ on the limit set Λ_θ, which is conformal, almost symmetric, and satisfies almost triangle inequality with bounded multiplicative error. We also discuss how this definition can be extended to non-symmetric θ at the end of the section.

Recall the definition of the Gromov product from Definition 2.3. The θ-Anosov property of Γ implies that any two distinct points in Λ_θ are in general position: if $\xi \neq \eta$ in Λ_θ, then $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$. Therefore the following premetric on Λ_θ is well-defined:

Definition 5.1. For $\xi, \eta \in \Lambda_\theta$, we set

$$d_\psi(\xi, \eta) = \begin{cases} e^{-\psi(\mathcal{G}^\theta(\xi,\eta))} & \text{if } \xi \neq \eta \\ 0 & \text{if } \xi = \eta. \end{cases} \tag{5.1}$$

We first observe the following Γ-conformal property of d_ψ:

Lemma 5.2. *For $\gamma \in \Gamma$ and $\xi, \eta \in \Lambda_\theta$, we have*

$$d_\psi(\gamma^{-1}\xi, \gamma^{-1}\eta) = e^{\frac{1}{2}\psi(\beta_\xi^\theta(e,\gamma) + \mathrm{i}(\beta_\eta^\theta(e,\gamma)))} d_\psi(\xi, \eta).$$

Proof. Let $\xi \neq \eta$, and $g \in G$ be such that $g^+ = \xi$ and $g^- = \eta$. Then for any $\gamma \in \Gamma$,

$$\begin{aligned} 2\mathcal{G}^\theta(\gamma^{-1}\xi, \gamma^{-1}\eta) &= \beta^\theta_{\gamma^{-1}\xi}(e, \gamma^{-1}g) + \mathrm{i}(\beta^\theta_{\gamma^{-1}\eta}(e, \gamma^{-1}g)) \\ &= 2\mathcal{G}^\theta(\xi, \eta) + \beta^\theta_\xi(\gamma, e) + \mathrm{i}(\beta^\theta_\eta(\gamma, e)) \\ &= 2\mathcal{G}^\theta(\xi, \eta) - \beta^\theta_\xi(e, \gamma) - \mathrm{i}(\beta^\theta_\eta(e, \gamma)). \end{aligned}$$

Now the claim follows from the definition of d_ψ. ☐

Recall that $\mathcal{G}^\theta(\xi, \eta) = \mathrm{i}(\mathcal{G}^\theta(\eta, \xi))$ for all $\xi, \eta \in \Lambda_\theta$. Hence if ψ is i-invariant, then d_ψ is symmetric. We have the following in general:

Proposition 5.3 (Metric-like properties of d_ψ)**.**

(1) *There exists $R = R(\psi) > 1$ such that for all $\xi, \eta \in \Lambda_\theta$,*

$$R^{-1} d_\psi(\eta, \xi) \leq d_\psi(\xi, \eta) \leq R\, d_\psi(\eta, \xi).$$

(2) *There exists $N = N(\psi) > 0$ such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda_\theta$,*

$$d_\psi(\xi_1, \xi_3) \leq N(d_\psi(\xi_1, \xi_2) + d_\psi(\xi_2, \xi_3)).$$

The second property was obtained in [37, Lemma 6.11] and the same proof can be repeated for a general θ in verbatim. The first property follows from Lemma 5.4 below. For $x \neq y$ in the Gromov boundary $\partial\Gamma$ and a bi-infinite geodesic $[x, y]$ in Γ, we denote by $\gamma_{x,y} \in [x, y]$ the nearest-point projection of the identity e to $[x, y]$ in $(\Gamma, \mathsf{d}_\Gamma)$, that is, $\gamma_{x,y} \in [x, y]$ is an element such that $\mathsf{d}_\Gamma(e, \gamma_{x,y}) = \inf\{\mathsf{d}_\Gamma(e, g) : g \in [x, y]\}$, which is coarsely well-defined. Recall the homeomorphism $f : \Gamma \cup \partial\Gamma \to \Gamma o \cup \Lambda_\theta$ from Theorem 3.8(4). The following was proved in [37, Lemma 6.6] for $\theta = \Pi$ and the same proof works for a general θ:

Lemma 5.4. *There exists $C_1 > 0$ such that for any $x \neq y \in \partial\Gamma$,*

$$\left\| \mathcal{G}^\theta(f(x), f(y)) - \frac{\mu_\theta(\gamma_{x,y}) + \mathrm{i}(\mu_\theta(\gamma_{x,y}))}{2} \right\| < C_1.$$

In particular, for $\xi \neq \eta \in \Lambda_\theta$, we have

$$\|\mathcal{G}^\theta(\xi, \eta) - \mathcal{G}^\theta(\eta, \xi)\| < 2C_1.$$

Symmetrization. Consider the following symmetrization of $\psi \in \mathfrak{a}_\theta^*$:

$$\bar{\psi} := \frac{\psi + \psi \circ \mathrm{i}}{2} \in \mathfrak{a}^*_{\theta \cup \mathrm{i}(\theta)}.$$

Since we are assuming $\theta = \mathrm{i}(\theta)$, we have $\bar{\psi} \in \mathfrak{a}_\theta^*$ as well. Since $\mathcal{L}_\theta$ is i-invariant, we have $\bar{\psi} > 0$ on $\mathcal{L}_\theta - \{0\}$. Lemma 5.4 implies that $d_{\bar{\psi}}$ and d_ψ are Lipschitz equivalent:

Proposition 5.5. *There exists $R \geq 1$ such that for any $\xi, \eta \in \Lambda_\theta$, we have*

$$R^{-1} d_\psi(\xi, \eta) \leq d_{\bar{\psi}}(\xi, \eta) \leq R d_\psi(\xi, \eta).$$

Proof. Since $\mathcal{G}^\theta(\eta, \xi) = \mathrm{i}(\mathcal{G}^\theta(\xi, \eta))$ for all $\eta \neq \xi$ in Λ_θ, it follows from Lemma 5.4 with the constant C_1 therein that

$$|\psi(\mathcal{G}^\theta(\xi, \eta)) - \bar{\psi}(\mathcal{G}^\theta(\xi, \eta))| = \frac{1}{2} |\psi(\mathcal{G}^\theta(\xi, \eta) - \mathcal{G}^\theta(\eta, \xi))| < \|\psi\| C_1.$$

It suffices to set $R = e^{\|\psi\| C_1}$ to finish the proof. $\square$

We also record the following Vitali covering type lemma which is a standard consequence of Proposition 5.3(2) (cf. [37]): here $B_\psi(\xi, r) = \{\eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r\}$.

Lemma 5.6. [37, Lemma 6.12] *There exists $N_0 = N_0(\psi) \geq 1$ satisfying the following: for any finite collection $B_\psi(\xi_1, r_1), \cdots, B_\psi(\xi_n, r_n)$ with $\xi_i \in \Lambda_\theta$ and $r_i > 0$ for $i = 1, \cdots, n$, there exists a disjoint subcollection $B_\psi(\xi_{i_1}, r_{i_1}), \cdots, B_\psi(\xi_{i_k}, r_{i_k})$ such that*

$$\bigcup_{i=1}^n B_\psi(\xi_i, r_i) \subset \bigcup_{j=1}^k B_\psi(\xi_{i_j}, N_0 r_{i_j}).$$

Remark 5.7. Recall that the canonical projection $p : \Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_\theta$ is a Γ-equivariant homeomorphism and that $\mathfrak{a}_\theta^* \subset \mathfrak{a}^*_{\theta \cup \mathrm{i}(\theta)}$. Using this homeomorphism, we can also define a function d_ψ on Λ_θ even when θ is not symmetric, so that $p : (\Lambda_{\theta \cup \mathrm{i}(\theta)}, d_\psi) \to (\Lambda_\theta, d_\psi)$ is an isometry:

$$d_\psi(\xi, \eta) := d_\psi(p^{-1}(\xi), p^{-1}(\eta))$$

for all $\xi, \eta \in \Lambda_\theta$. In this regard, the above discussion is still valid without the symmetric hypothesis on θ.

6. Compatibility of shadows and d_ψ-balls

As before, let Γ be a θ-Anosov subgroup of a connected semisimple real algebraic group G. We fix a word metric d_Γ on Γ. Fix a linear form

$$\psi \in \mathfrak{a}_\theta^*$$

which is positive on $\mathcal{L} - \{0\}$ and $\psi = \psi \circ \mathrm{i}$. Recall the premetric d_ψ on Γo defined in (4.1) and the conformal premetric d_ψ on Λ_θ defined by (5.1).

Lemma 6.1. *Both $(\Gamma o, \mathsf{d}_\psi)$ and (Λ_θ, d_ψ) are symmetric.*

Proof. For $g_1, g_2 \in G$, we have

$$\mathsf{d}_\psi(g_1 o, g_2 o) = \psi(\mu(g_1^{-1} g_2)) = \psi \circ \mathrm{i}(\mu(g_2^{-1} g_1)) = \mathsf{d}_\psi(g_2 o, g_1 o).$$

The second claim follows similarly since $\mathcal{G}^{\theta \cup \mathrm{i}(\theta)}(\xi, \eta) = \mathrm{i}\, \mathcal{G}^{\theta \cup \mathrm{i}(\theta)}(\eta, \xi)$ for all $\xi, \eta \in \mathcal{F}_{\theta \cup \mathrm{i}(\theta)}$ in general position. □

Shadows play a basic role in studying the metric property of (Λ_θ, d_ψ) in relation with the geometry of the symmetric space X, as in the original work of Sullivan. We recall the definition of shadows in $\mathcal{F}_\theta$. For $p, q \in X$, the shadow $O_R^\theta(p, q)$ of the Riemannian ball $B(q, R)$ viewed from p is defined as

$$O_R^\theta(p, q) = \{g P_\theta \in \mathcal{F}_\theta : g \in G,\ go = p,\ d(q, gA^+ o) < R\}.$$

We refer to [32] and [33] for basic properties of these shadows.

The main technical ingredient of this paper is the following theorem which says that shadows in Λ_θ are comparable with d_ψ-balls.

Theorem 6.2. *Let $\psi \in \mathfrak{a}_\theta^*$ be such that $\psi > 0$ on $\mathcal{L} - \{0\}$ and $\psi = \psi \circ \mathrm{i}$. Then there exist constants $c, R_0 > 0$ such that for any $R > R_0$, there exists $c' = c'_R > 0$ so that the following holds: for any $\xi \in \Lambda_\theta$ and any $g \in \Gamma$ on a geodesic ray $[e, \xi]$ in Γ, we have*

$$B_\psi(\xi, c e^{-\mathsf{d}_\psi(o, go)}) \subset O_R^\theta(o, go) \cap \Lambda_\theta \subset B_\psi(\xi, c' e^{-\mathsf{d}_\psi(o, go)}). \tag{6.1}$$

Since the proof of this theorem is quite lengthy, we will prove the first inclusion in this section and the second inclusion in the next section. The rest of this section is devoted to the proof of the first inclusion. In view of Remark 5.7, we assume that

$$\theta = \mathrm{i}(\theta).$$

Strictly speaking, d_ψ is not a metric on the Γ-orbit Γo. Nevertheless, we will still employ terminologies for the metric space on $(\Gamma o, \mathsf{d}_\psi)$ for convenience. For instance, for a subset $B \subset \Gamma o$, $\mathsf{d}_\psi(go, B) = \inf_{ho \in B} \mathsf{d}_\psi(go, ho)$ and the R-neighborhood of B is given by $\{go \in \Gamma o : \mathsf{d}_\psi(go, B) < R\}$, etc.

Two main ingredients of the proof of the first inclusion of (6.1) are the following, which allow us to treat $(\Gamma o, \mathsf{d}_\psi)$ almost like a Gromov hyperbolic space:

1. $(\Gamma o, \mathsf{d}_\psi)$ satisfies a triangle inequality up to an additive error (Theorem 4.1);
2. the ψ-Gromov product $\psi(\mathcal{G}(\xi, \eta))$ is equal to the premetric $\mathsf{d}_\psi(o, [\xi, \eta] o)$ up to an additive error (Proposition 6.7).

In the rank one case, the property (2) is a well-known consequence of a uniform thin-triangle property and the Morse lemma of the rank one symmetric space. Higher rank symmetric spaces have neither of these properties. Our proof of (2) is based on the KLP Morse lemma using diamonds as well as a uniform thin-triangle property of the orbit $(\Gamma o, \mathsf{d}_\psi)$.

We begin with the following:

Proposition 6.3. *The orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (\Gamma o, \mathsf{d}_\psi)$, $\gamma \mapsto \gamma o$, is a quasi-isometry, i.e., there exists $Q_\psi \geq 1$ such that for any $\gamma_1, \gamma_2 \in \Gamma$,*

$$Q_\psi^{-1} \cdot \mathsf{d}_\Gamma(\gamma_1, \gamma_2) - Q_\psi \leq \mathsf{d}_\psi(\gamma_1 o, \gamma_2 o) \leq Q_\psi \cdot \mathsf{d}_\Gamma(\gamma_1, \gamma_2) + Q_\psi.$$

In particular, the images of geodesic triangles in Γ under the orbit map are uniformly thin, that is, there exists $T_\psi > 0$ such that for any $\xi_1, \xi_2, \xi_3 \in \Gamma \cup \partial\Gamma$, the image $[\xi_1, \xi_3]o$ is contained in the T_ψ-neighborhood of $([\xi_1, \xi_2] \cup [\xi_2, \xi_3])o$ with respect to d_ψ.

Proof. The second part follows since $(\Gamma, \mathsf{d}_\Gamma)$ is a Gromov hyperbolic space (Theorem 3.8(1)), and hence it has a uniform thin-triangle property which is a quasi-isometry invariance. Since the orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (\Gamma o, d)$ is a quasi-isometry (Theorem 3.8(3)), the first part of the above proposition follows from the following claim that the identity map $(\Gamma o, d) \to (\Gamma o, \mathsf{d}_\psi)$ is a quasi-isometry: there exists $C_\psi \geq 1$ such that for all $\gamma_1, \gamma_2 \in \Gamma$, we have

$$C_\psi^{-1} d(\gamma_1 o, \gamma_2 o) - C_\psi \leq \mathsf{d}_\psi(\gamma_1 o, \gamma_2 o) \leq C_\psi d(\gamma_1 o, \gamma_2 o) + C_\psi. \tag{6.2}$$

We can take a cone $\mathcal{C}$ containing a neighborhood of $\mathcal{L} - \{0\}$ in $\mathfrak{a}^+$ such that $\psi > 0$ on $\mathcal{C} - \{0\}$. Hence we can chose $C_1 > 1$ so that

$$C_1^{-1} < \min_{u \in \mathcal{C}, \|u\|=1} \psi(u) \leq \max_{u \in \mathcal{C}, \|u\|=1} \psi(u) < C_1.$$

On the other hand, $\mu(\gamma) \in \mathcal{C}$ for all but finitely many $\gamma \in \Gamma$ (Lemma 4.4), and hence $C_2 := \max\{|\psi(\mu(\gamma))| : \mu(\gamma) \notin \mathcal{C}\} < \infty$. If we set $C = C_1 + C_2$, then

$$C^{-1}\|\mu(\gamma)\| - C \leq \psi(\mu(\gamma)) \leq C\|\mu(\gamma)\| + C.$$

Since both d and d_ψ are left Γ-invariant, this implies the claim. □

We use the Morse property to obtain that the image of a geodesic ray under the orbit map has a uniform progression:

Lemma 6.4 (Uniform progression lemma)**.** *For any $r > 0$, there exists $n_r > 0$ such that for any geodesic ray $\{\gamma_0 = e, \gamma_1, \gamma_2, \cdots\}$ in $(\Gamma, \mathsf{d}_\Gamma)$,*

$$\mathsf{d}_\psi(o, \gamma_{i+n} o) \geq \mathsf{d}_\psi(o, \gamma_i o) + r$$

for all $i \in \mathbb{N}$ and all $n \geq n_r$.

Proof. Fix $r > 0$. By Theorem 4.13, there exist a cone $\mathcal{C} \subset \mathfrak{a}^+$ and $B, D_0 \geq 0$ so that for all $n \geq B$ and $i \geq 0$, the sequence $o, \gamma_1 o, \cdots, \gamma_{i+n} o$ is contained in the D_0-neighborhood of the diamond $\Diamond_{\mathcal{C}}(o, \gamma_{i+n} o)$ in (X, d). We may also assume that $\psi > 0$ on $\mathcal{C} - \{0\}$ as $\mathcal{C}$ can be arbitrarily close to $\mathcal{L}$. For each $i \geq 0$, choose a point $x_i \in \Diamond_{\mathcal{C}}(o, \gamma_{i+n} o)$ which is D_0-close to $\gamma_i o$. Applying Lemma 4.10, we obtain that

$$\mathsf{d}_\psi(o, x_i) + \mathsf{d}_\psi(x_i, \gamma_{i+n} o) = \mathsf{d}_\psi(o, \gamma_{i+n} o). \tag{6.3}$$

Since the orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (\Gamma o, d)$ is a Q_ψ-quasi-isometry by Proposition 6.3, we get that for all $i \geq 0$,

$$\mathsf{d}_\psi(\gamma_i o, \gamma_{i+n} o) \geq Q_\psi^{-1} \mathsf{d}_\Gamma(\gamma_i, \gamma_{i+n}) - Q_\psi = Q_\psi^{-1} n - Q_\psi. \tag{6.4}$$

By applying Lemma 2.1 to a compact subset $\{g \in G : \|\mu(g)\| \le D_0\}$, we have

$$\begin{aligned} &|\mathsf{d}_\psi(o, x_i) - \mathsf{d}_\psi(o, \gamma_i o)| \le C \quad \text{and} \\ &|\mathsf{d}_\psi(x_i, \gamma_{i+n} o) - \mathsf{d}_\psi(\gamma_i o, \gamma_{i+n} o)| \le C \end{aligned} \tag{6.5}$$

where C depends only on D_0 and $\|\psi\|$. Putting (6.3), (6.4), and (6.5) together, we get

$$\begin{aligned} \mathsf{d}_\psi(o, \gamma_{i+n} o) &= \mathsf{d}_\psi(o, x_i) + \mathsf{d}_\psi(x_i, \gamma_{i+n} o) \\ &\ge \mathsf{d}_\psi(o, \gamma_i o) + \mathsf{d}_\psi(\gamma_i o, \gamma_{i+n} o) - 2C \\ &\ge \mathsf{d}_\psi(o, \gamma_i o) + (Q_\psi^{-1} n - Q_\psi) - 2C. \end{aligned}$$

Hence setting $n_r = B + Q_\psi(r + 2C + Q_\psi)$ finishes the proof. □

Lemma 6.5 (Small inscribed triangle)**.** *There exists $C > 0$ satisfying the following property: Let $[\xi, \eta]$ be a bi-infinite geodesic in $(\Gamma, \mathsf{d}_\Gamma)$. If γo is the nearest-point projection of o to $[\xi, \eta]o$ in the d_ψ-metric, i.e., $\gamma \in [\xi, \eta]$ is such that $\mathsf{d}_\psi(o, \gamma o) = \mathsf{d}_\psi(o, [\xi, \eta]o)$, then there exist $u \in [e, \xi]$ and $v \in [e, \eta]$ so that $\{uo, vo, \gamma o\}$ has d_ψ-diameter less than C.*

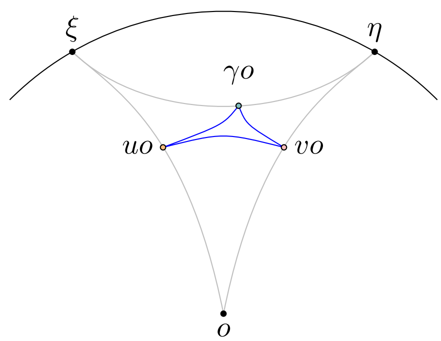

FIGURE 3. Small inscribed triangle

Proof. Recall from Proposition 6.3 that there exists $T_\psi > 0$ so that every triangle in Γo, obtained as the image of a geodesic triangle in $(\Gamma, \mathsf{d}_\Gamma)$ under the orbit map, is T_ψ-thin in the d_ψ-metric. By the T_ψ-thinness of $(\Gamma o, \mathsf{d}_\psi)$, we have either $\mathsf{d}_\psi(\gamma o, [e, \xi]o) \le T_\psi$ or $\mathsf{d}_\psi(\gamma o, [e, \eta]o) \le T_\psi$. We will assume the latter case; the other case can be treated similarly. We write $[e, \eta] = \{v_i\}_{i \ge 0}$. We then can choose j so that $j = \min\{i \ge 0 : \mathsf{d}_\psi(\gamma o, v_i o) \le T_\psi + D\}$ where D is given in Theorem 4.1. Let $n' = n_{3T_\psi + 3D}$ be the constant from Lemma 6.4. If $j < n'$, then we set $u = v = e$ and note that

$$\begin{aligned} \mathsf{d}_\psi(\gamma o, o) &\le \mathsf{d}_\psi(\gamma o, v_j o) + \mathsf{d}_\psi(v_j o, o) + D \\ &\le D_1 := T_\psi + (n' + 1)Q_\psi + 2D \end{aligned} \tag{6.6}$$

where Q_ψ is given by Proposition 6.3. Hence the triangle $\{uo, vo, \gamma o\} = \{o, o, \gamma o\}$ has d_ψ-diameter at most D_1.

Now suppose that $j > n'$. We claim that

$$\mathsf{d}_\psi(v_{j-n'}o, [\xi, \eta]o) > T_\psi.$$

Indeed, otherwise, $\mathsf{d}_\psi(v_{j-n'}o, \gamma' o) \leq T_\psi$ for some $\gamma' \in [\xi, \eta]$, and hence we have

$$\begin{aligned}
\mathsf{d}_\psi(o, \gamma' o) &\leq \mathsf{d}_\psi(o, v_{j-n'}o) + \mathsf{d}_\psi(v_{j-n'}o, \gamma' o) + D \\
&\leq (\mathsf{d}_\psi(o, v_j o) - 3T_\psi - 3D) + T_\psi + D \\
&= \mathsf{d}_\psi(o, v_j o) - 2T_\psi - 2D \\
&\leq \mathsf{d}_\psi(o, v_j o) - \mathsf{d}_\psi(\gamma o, v_j o) - T_\psi - D \\
&\leq \mathsf{d}_\psi(o, \gamma o) - T_\psi,
\end{aligned}$$

where the first and the last inequalities follow from Theorem 4.1 and the second is from Lemma 6.4. This yields a contradiction to the minimality of $\mathsf{d}_\psi(o, \gamma o)$, proving the claim.

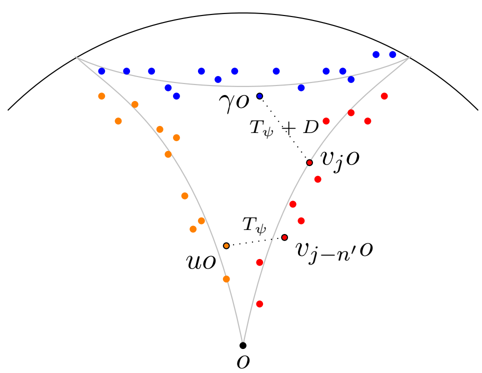

FIGURE 4. Choice of γo, $v_j o$, $v_{j-n'}o$ and $u_k o$

Since the triangle consisting of the sides $[\xi, \eta]o$, $[e, \xi]o$, and $[e, \eta]o = \{v_i o\}_{i \geq 0}$ is T_ψ-thin, the above claim implies that $v_{j-n'}o$ lies in the T_ψ-neighborhood of $[e, \xi]o$. Hence there exists $u \in [e, \xi]$ such that $\mathsf{d}_\psi(v_{j-n'}o, uo) \leq T_\psi$ (see Figure 4). Since $\mathsf{d}_\psi(v_j, v_{j-n'}) \leq Q_\psi n' + Q_\psi$, we have so far obtained

- $\mathsf{d}_\psi(\gamma o, v_j o) \leq T_\psi + D$;
- $\mathsf{d}_\psi(v_j o, uo) \leq Q_\psi n' + Q_\psi + T_\psi + D$;
- $\mathsf{d}_\psi(\gamma o, uo) \leq Q_\psi n' + Q_\psi + 2T_\psi + 3D$.

Therefore the triangle $\{uo, v_j o, \gamma o\}$ has d_ψ-diameter at most $D_2 = Q_\psi n' + Q_\psi + 2T_\psi + 3D$. It remains to set $C = \max(D_1, D_2)$. $\square$

The following was shown for $\theta = \Pi$ in [37, Lemma 5.7] which directly implies the statement for general θ:

Lemma 6.6. *There exists $\kappa > 0$ such that for any $g, h \in G$ and $R > 0$, we have*

$$\sup_{\xi \in O_R^\theta(go,ho)} \|\beta_\xi^\theta(go, ho) - \mu_\theta(g^{-1}h)\| \le \kappa R.$$

We now prove that the ψ-Gromov product $\psi(\mathcal{G}(\xi, \eta))$ behaves like the distance $\mathsf{d}_\psi(o, [\xi, \eta]o)$ up to an additive error:

Proposition 6.7 (Comparison between ψ-Gromov product and d_ψ-distance)**.** *There exists $C_1 > 0$ such that for any $\xi \ne \eta \in \Lambda_\theta = \partial\Gamma$, we have*

$$|\psi(\mathcal{G}^\theta(\xi, \eta)) - \mathsf{d}_\psi(o, [\xi, \eta]o)| \le C_1.$$

Proof. Let $\gamma \in [\xi, \eta]$ be such that $\mathsf{d}_\psi(o, \gamma o) = \mathsf{d}_\psi(o, [\xi, \eta]o)$. Consider geodesic rays $[e, \xi]$ and $[e, \eta]$ in $(\Gamma, \mathsf{d}_\Gamma)$. Let $k, \ell \in K$ and $h \in G$ be such that $kP_\theta = \xi$, $\ell P_\theta = \eta$, $hP_\theta = \xi$ and $hw_0P_\theta = \eta$. For the constant D_0 given by Theorem 4.13, we have

$$\begin{aligned} &\sup_{u \in [e,\xi]} d(uo, kM_\theta A^+ o) \le D_0; \\ &\sup_{v \in [e,\eta]} d(vo, \ell M_\theta A^+ o) \le D_0; \\ &\sup_{g \in [\xi,\eta]} d(go, hM_\theta A o) \le D_0. \end{aligned} \tag{6.7}$$

Since $\gamma \in [\xi, \eta]$ by the choice, the third inequality implies that $d(\gamma o, hM_\theta Ao) \le D_0$. We may assume that h satisfies that $d(ho, \gamma o) \le D_0$, by replacing h with an element of $hM_\theta A$ if necessary.

We first claim that for some uniform $R > 0$ depending only on Γ and ψ,

$$\xi, \eta \in O_R^\theta(o, \gamma o).$$

To show the claim, let $C > 0$ be the constant given by Lemma 6.5 and choose $u \in [e, \xi]$ and $v \in [e, \eta]$ so that the triangle $\{uo, vo, \gamma o\}$ has d_ψ-diameter smaller than C (see Figure 5). Hence, for the constant $C' := C_\psi(C + C_\psi)$, where C_ψ is given in (6.2), the Riemannian diameter of the triangle $\{uo, vo, \gamma o\}$ is less than C'. It then follows from the first two inequalities of (6.7) that

$$d(\gamma o, kM_\theta A^+ o) < D_0 + C' \quad \text{and} \quad d(\gamma o, \ell M_\theta A^+ o) < D_0 + C'.$$

Since $kP_\theta = \xi$ and $\ell P_\theta = \eta$, we have

$$\xi, \eta \in O_{D_0 + C'}^\theta(o, \gamma o),$$

showing the claim with $R = D_0 + C'$

Therefore by Lemma 6.6, we get

$$\|\beta_\xi^\theta(o, \gamma o) - \mu_\theta(\gamma)\| \le \kappa R \quad \text{and} \quad \|\beta_\eta^\theta(o, \gamma o) - \mu_\theta(\gamma)\| \le \kappa R.$$

Since $\beta_\xi^\theta(o, \gamma o) = \beta_\xi^\theta(o, ho) + \beta_\xi^\theta(ho, \gamma o)$ and $\|\beta_\xi^\theta(ho, \gamma o)\| \le d(ho, \gamma o) \le D_0$, we have

$$\|\beta_\xi^\theta(o, ho) - \mu_\theta(\gamma)\| \le \kappa R + D_0$$

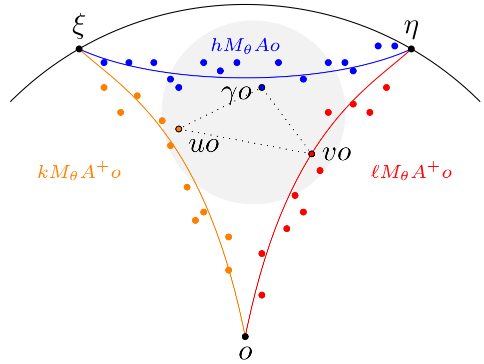

FIGURE 5. The dotted triangle is of diameter less than C and the gray ball has radius R.

and similarly

$$\|\beta_\eta^\theta(o, ho) - \mu_\theta(\gamma)\| \leq \kappa R + D_0.$$

Recalling the definition $\mathcal{G}^\theta(\xi, \eta) = \frac{1}{2}(\beta_\xi^\theta(o, ho) + \mathrm{i}(\beta_\eta^\theta(o, ho)))$, and using $\psi = \psi \circ \mathrm{i}$, we obtain that

$$|\psi(\mathcal{G}^\theta(\xi, \eta)) - \mathsf{d}_\psi(o, \gamma o)| \leq \|\psi\|(\kappa R + D_0),$$

as desired. □

We are now ready to prove the first inclusion in Theorem 6.2 which we formulate again as follows:

Proposition 6.8. *There exist constants $c, R_0 > 0$ such that for any $\xi \in \Lambda_\theta$ and $g \in [e, \xi]$ in Γ, we have*

$$B_\psi(\xi, ce^{-\mathsf{d}_\psi(o,go)}) \subset O_{R_0}^\theta(o, go) \cap \Lambda_\theta. \tag{6.8}$$

Proof. Let $C_1, D > 0$ be the constants given by Proposition 6.7 and Theorem 4.1 respectively. Recall the constant T_ψ in Proposition 6.3; the image of any geodesic triangle in $(\Gamma, \mathsf{d}_\Gamma)$ under the orbit map, is T_ψ-thin in the d_ψ-metric. We now claim that (6.8) holds with $c := e^{-(2T_\psi + C_1 + D)}$. Fix $\xi \in \Lambda_\theta$ and an element $g \in [e, \xi]$. Let $\eta \in B_\psi(\xi, ce^{-\mathsf{d}_\psi(o,go)})$, that is,

$$\psi(\mathcal{G}^\theta(\xi, \eta)) > \mathsf{d}_\psi(o, go) + 2T_\psi + C_1 + D. \tag{6.9}$$

Let $\gamma \in [\xi, \eta]$ be chosen so that $\mathsf{d}_\psi(o, \gamma o) = \mathsf{d}_\psi(o, [\xi, \eta]o)$. By Proposition 6.7, we have

$$\mathsf{d}_\psi(o, \gamma o) \geq \psi(\mathcal{G}^\theta(\xi, \eta)) - C_1.$$

Hence by (6.9),

$$\mathsf{d}_\psi(o, \gamma o) > \mathsf{d}_\psi(o, go) + 2T_\psi + D. \tag{6.10}$$

Let $g' \in [\xi, \eta]$ be such that $\mathsf{d}_\psi(go, g'o) = \mathsf{d}_\psi(go, [\xi, \eta]o)$. By Theorem 4.1, we also have

$$\mathsf{d}_\psi(o, \gamma o) \le \mathsf{d}_\psi(o, g'o) \le \mathsf{d}_\psi(o, go) + \mathsf{d}_\psi(go, [\xi, \eta]o) + D.$$

Together with (6.10), this implies

$$\mathsf{d}_\psi(go, [\xi, \eta]o) > 2T_\psi. \tag{6.11}$$

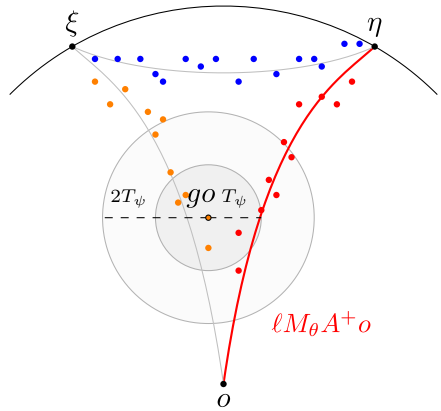

FIGURE 6. go is far from $[\xi, \eta]o$ and hence close to $[e, \eta]o$; so η lies in the shadow $O^\theta_{T'+D_0}(o, go)$.

Since the triangle $[e, \xi]o \cup [\xi, \eta]o \cup [e, \eta]o$ is T_ψ-thin in d_ψ-metric, go is contained in the T_ψ-neighborhood of $[\xi, \eta]o \cup [e, \eta]o$. Since $\mathsf{d}_\psi(go, [\xi, \eta]o) > 2T_\psi$ by (6.11), we must have $\mathsf{d}_\psi(go, [e, \eta]o) \le T_\psi$ (see Figure 6). For the constant $T' := C_\psi(T_\psi + C_\psi)$ where C_ψ is as in (6.2), we have

$$d(go, [e, \eta]o) \le T'.$$

With the constant D_0 given in Theorem 4.13, there exists $\ell \in K$ so that $\ell P_\theta = \eta$ and $[e, \eta]o$ is contained in the D_0-neighborhood of $\ell M_\theta A^+ o$ in the Riemannian metric d. This implies that

$$\eta \in O^\theta_{T'+D_0}(o, go) \cap \Lambda_\theta.$$

This completes the proof with $R_0 = T' + D_0$. $\square$

7. SHADOWS INSIDE BALLS: THE SECOND INCLUSION IN THEOREM 6.2

We continue the setup from section 6. Hence $\mathrm{i}(\theta) = \theta$ and $\psi \in \mathfrak{a}^*_\theta$ is a linear form such that $\psi > 0$ on $\mathcal{L} - \{0\}$ and $\psi = \psi \circ \mathrm{i}$. In this section, we prove the second inclusion of Theorem 6.2, which can be stated as follows:

Proposition 7.1. *For any $r > 0$, there exists $c' = c'_r > 0$ such that for any $\xi \in \Lambda_\theta$ and any $g \in [e, \xi]$ in Γ, we have*

$$O^\theta_r(o, go) \cap \Lambda_\theta \subset B_\psi(\xi, c' e^{-\mathsf{d}_\psi(o, go)}).$$

In addition to the coarse triangle inequality of d_ψ (Theorem 4.1) and the uniform progression lemma (Lemma 6.4), we will use the property that the shadows in $(\Gamma, \mathsf{d}_\Gamma)$ are comparable to shadows in Λ_θ (Proposition 7.2) and that the half-spaces spanned by shadows of balls in $(\Gamma, \mathsf{d}_\Gamma)$ stay deeper than the balls from the viewpoints (see Figure 7 and Lemma 7.4).

In the Gromov hyperbolic space $(\Gamma, \mathsf{d}_\Gamma)$, for $R > 0$ and $\gamma_1, \gamma_2 \in \Gamma$, the shadow $O_R^\Gamma(\gamma_1, \gamma_2)$ is defined as the set of all $\xi \in \partial\Gamma$ such that a geodesic ray $[\gamma_1, \xi]$ intersects the R-ball centered at γ_2:

$$O_R^\Gamma(\gamma_1, \gamma_2) = \{\xi \in \partial\Gamma : \mathsf{d}_\Gamma(\gamma_2, [\gamma_1, \xi]) < R\}.$$

Clearly, shadows are Γ-equivariant in the sense that for any $\gamma \in \Gamma$, we have $\gamma O_R^\Gamma(\gamma_1, \gamma_2) = O_R^\Gamma(\gamma\gamma_1, \gamma\gamma_2)$.

The following proposition states that shadows in $\partial\Gamma$ and shadows in Λ_θ are compatible via the boundary map $f : \partial\Gamma \to \Lambda_\theta$: recall that the orbit map $(\Gamma, \mathsf{d}_\Gamma) \to (\Gamma o, d)$ is a Q-quasi-isometry for some $Q \geq 1$ (Theorem 3.8(3)) and let $R_0 := Q + D_0 + 1$ where D_0 is given in Theorem 4.13.

Proposition 7.2. *For any $R > R_0$, there exists $R_1, R_2 > 0$ such that for any $\gamma_1, \gamma_2 \in \Gamma$,*

$$f(O_{R_1}^\Gamma(\gamma_1, \gamma_2)) \subset O_R^\theta(\gamma_1 o, \gamma_2 o) \cap \Lambda_\theta \subset f(O_{R_2}^\Gamma(\gamma_1, \gamma_2)).$$

In proving this proposition, we will also need to consider shadows whose viewpoints are on the boundary $\mathcal{F}_\theta$. For $\eta \in \mathcal{F}_\theta$, $p \in X$, and $R > 0$, the θ-shadow $O_R^\theta(\eta, p)$ is defined as follows:

$$O_R^\theta(\eta, p) = \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ gw_0P_\theta = \eta,\ d(p, go) < R\}.$$

We will need the following proposition on continuity of shadows:

Proposition 7.3 (Continuity of shadows on viewpoints, [32, Proposition 3.4])**.** *Let $p \in X$, $\eta \in \mathcal{F}_\theta$ and $r > 0$. If a sequence $q_i \in X$ converges to η as $i \to \infty$ as in Definition 2.2, then for any $0 < \varepsilon < r$, we have*

$$O_{r-\varepsilon}^\theta(\eta, p) \subset O_r^\theta(q_i, p) \subset O_{r+\varepsilon}^\theta(\eta, p) \quad \textit{for all large } i \geq 1. \tag{7.1}$$

Proof of Proposition 7.2. Let $R > R_0$. By the Γ-equivariance of f as well as of shadows, we may assume $\gamma_1 = e$ and write $\gamma_2 = \gamma$. By applying Theorem 4.13(2), we get that for any $\xi \in \partial\Gamma$ and $k \in K$ with $kP_\theta = f(\xi)$, the image $[e, \xi]o$ is contained in the D_0-neighborhood of $kM_\theta(\exp \mathcal{C})o \subset kM_\theta A^+ o$ in the symmetric space (X, d). Since $R > R_0 = Q + D_0 + 1$, we can choose $R_1 > 0$ so that $QR_1 + Q + D_0 < R$. Now if $\xi \in O_{R_1}^\Gamma(e, \gamma)$, and hence $[e, \xi]$ intersects the ball $\{g \in \Gamma : \mathsf{d}_\Gamma(\gamma, g) < R_1\}$, then $kM_\theta A^+ o$ intersects the $QR_1 + Q + D_0$-neighborhood of γo, and hence the R-neighborhood of γo. Therefore $f(\xi) \in O_R^\theta(o, \gamma o)$. This shows the first inclusion.

To prove the second inclusion, suppose that the claim does not hold for some $R > R_0$. Then for each $i \geq 1$, there exists $\gamma_i \in \Gamma$ such that

$$O_R^\theta(o, \gamma_i o) \cap \Lambda_\theta \not\subset f(O_i^\Gamma(e, \gamma_i));$$

in other words, there exists $x_i \in \partial\Gamma - O_i^\Gamma(e, \gamma_i)$ such that $f(x_i) \in O_R^\theta(o, \gamma_i o)$. By the Γ-equivariance of f, it follows that

$$\gamma_i^{-1} x_i \notin O_i^\Gamma(\gamma_i^{-1}, e) \quad \text{and} \quad f(\gamma_i^{-1} x_i) \in O_R^\theta(\gamma_i^{-1} o, o) \quad \text{for all } i \geq 1.$$

After passing to a subsequence, we may assume that $\gamma_i^{-1} \to y \in \partial\Gamma$ and $\gamma_i^{-1} x_i \to x$ as $i \to \infty$. By Theorem 3.8(4), we deduce $\gamma_i^{-1} o \to f(y)$ as $i \to \infty$. Applying Proposition 7.3 to $q_i = \gamma_i^{-1} o$, $p = o$ and $\eta = f(y)$, we have for some $\varepsilon > 0$ that

$$O_R^\theta(\gamma_i^{-1} o, o) \subset O_{R+\varepsilon/2}^\theta(f(y), o) \quad \text{for all } i \geq 1.$$

Since $f(\gamma_i^{-1} x_i) \in O_R^\theta(\gamma_i^{-1} o, o)$ for all $i \geq 1$ and $f(\gamma_i^{-1} x_i)$ converges to $f(x)$ as $i \to \infty$, we have

$$f(x) \in O_{R+\varepsilon}^\theta(f(y), o).$$

This implies that $f(x)$ is in general position with $f(y)$, i.e., $(f(x), f(y)) \in \mathcal{F}_\theta^{(2)}$, and in particular $f(x) \neq f(y)$. On the other hand, since $\gamma_i^{-1} x_i \notin O_i^\Gamma(\gamma_i^{-1}, e)$ for all $i \geq 1$, the sequence of geodesics $[\gamma_i^{-1} x_i, \gamma_i^{-1}]$ escapes any large ball centered at e. This implies that two sequences $\gamma_i^{-1} x_i$ and γ_i^{-1} must have the same limit, and hence $x = y$ which is a contradiction. Therefore the claim follows. $\square$

The Gromov product in $(\Gamma, \mathsf{d}_\Gamma)$ is defined as follows: for $\alpha, \beta, \gamma \in \Gamma$,

$$(\alpha, \beta)_\gamma = \frac{1}{2}\left(\mathsf{d}_\Gamma(\alpha, \gamma) + \mathsf{d}_\Gamma(\beta, \gamma) - \mathsf{d}_\Gamma(\alpha, \beta)\right)$$

and for $x, y \in \partial\Gamma$,

$$(x, y)_\gamma = \sup \liminf_{i,j \to \infty} (x_i, y_j)_\gamma$$

where the supremum is taken over all sequences $\{x_i\}, \{y_j\}$ in Γ such that $\lim_{i\to\infty} x_i = x$ and $\lim_{j\to\infty} y_j = y$. The Gromov product for a pair of a point in Γ and a point in $\partial\Gamma$ is defined similarly. The Gromov product $(x, y)_\gamma$ is known to measure distance from γ and to a geodesic $[x, y]$ up to a uniform additive error (see [11] for basic properties of Gromov hyperbolic spaces).

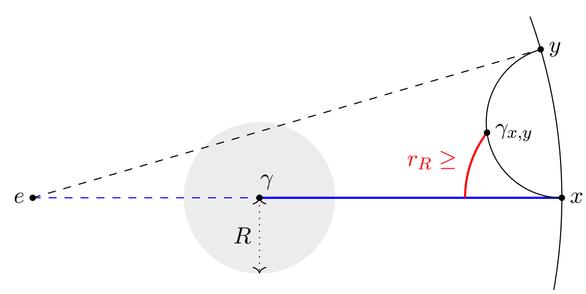

FIGURE 7. Pictorial description of Lemma 7.4.

The following lemma says that the half-space spanned by the shadow is opposite to the light; more precisely, for any $x \in \partial\Gamma$ and $\gamma \in [e, x]$, the half-space spanned by all geodesics connecting x and $O_R^\Gamma(e,\gamma)$ lies farther than γ, viewed from e:

Lemma 7.4. *Given $R > 0$, there exists $r = r_R > 0$ such that for any $x \in \partial\Gamma$, $\gamma \in [e,x]$, and $y \in O_R^\Gamma(e,\gamma)$, we have*

$$\mathsf{d}_\Gamma(\gamma_{x,y}, [\gamma, x]) \le r$$

where $\gamma_{x,y} \in [x,y]$ denotes the nearest-point projection of e to a geodesic $[x,y]$.

Proof. Let $[e,x] = \{\gamma_i\}_{i\ge 0}$. We fix $\gamma := \gamma_i$ and $y \in O_R^\Gamma(e,\gamma)$. In terms of the Gromov product, we have $(e,y)_\gamma < R + \delta/2$ for some uniform $\delta > 0$ depending only on Γ. On the other hand, the hyperbolicity of Γ also implies that we can take δ large enough so that

$$(e,y)_\gamma \ge \min\{(e,\gamma_{x,y})_\gamma, (\gamma_{x,y}, y)_\gamma\} - \delta/2$$

and that every geodesic triangle in $\Gamma \cup \partial\Gamma$ is δ-thin. Therefore

$$\min\{(e,\gamma_{x,y})_\gamma, (\gamma_{x,y}, y)_\gamma\} < R + \delta.$$

First consider the case when $(\gamma_{x,y}, y)_\gamma < R + \delta$. Then for some constant δ_1 depending on $R+\delta$, there exists $\gamma' \in [\gamma_{x,y}, y]$ such that $\mathsf{d}_\Gamma(\gamma',\gamma) < \delta_1$. Consider the geodesic triangle with vertices x, γ, γ'. Since this triangle is δ-thin and $\gamma_{x,y} \in [x,\gamma']$, the δ-neighborhood of $\gamma_{x,y}$ intersects $[x,\gamma] \cup [\gamma,\gamma']$. Hence it follows from $\mathsf{d}_\Gamma(\gamma,\gamma') < \delta_1$ that the $(\delta+\delta_1)$-neighborhood of $\gamma_{x,y}$ intersects the geodesic $[x,\gamma]$. Namely,

$$\mathsf{d}_\Gamma(\gamma_{x,y}, [\gamma, x]) \le \delta + \delta_1.$$

Now consider the case that $(e,\gamma_{x,y})_\gamma < R + \delta$. Since $\gamma_{x,y}$ is the nearest-point projection of e to $[x,y]$, there exists a constant δ_2 depending only on Γ such that the δ_2-neighborhood of $\gamma_{x,y}$ intersects both geodesic rays $[e,x]$ and $[e,y]$. In particular, there exists $\gamma_k \in [e,x]$ such that $\mathsf{d}_\Gamma(\gamma_{x,y},\gamma_k) < \delta_2$. This implies

$$(e,\gamma_k)_\gamma \le (e,\gamma_{x,y})_\gamma + \mathsf{d}_\Gamma(\gamma_{x,y},\gamma_k) < R + \delta + \delta_2.$$

Since both $\gamma = \gamma_i$ and γ_k lie on the geodesic $[e,x]$, this implies that $k \ge i - (R+\delta+\delta_2)$. Let j be the unique integer such that $k + R + \delta + \delta_2 \le j \le k + R + \delta + \delta_2 + 1$. Note that since $k \ge i - (R+\delta+\delta_2)$, we have $j \ge i$, and hence $\gamma_j \in [\gamma, x]$. Then

$$\begin{aligned}\mathsf{d}_\Gamma(\gamma_{x,y}, [\gamma, x]) &\le \mathsf{d}_\Gamma(\gamma_{x,y}, \gamma_j) \\ &\le \mathsf{d}_\Gamma(\gamma_{x,y}, \gamma_k) + \mathsf{d}_\Gamma(\gamma_k, \gamma_j) \\ &\le \delta_2 + (j-k) \le R + \delta + 2\delta_2 + 1.\end{aligned}$$

Therefore it remains to set $r = R + \delta + \delta_1 + 2\delta_2 + 1$. □

Now we are ready to prove:

Proof of Proposition 7.1. Let $\xi \in \Lambda_\theta = \partial\Gamma$ and $g \in [e, \xi]$ in Γ. Fix $r > 0$, and let $\eta \in O_r^\theta(o, go) \cap \Lambda_\theta$ distinct from ξ. We will continue to use the convention of identifying Λ_θ and $\partial\Gamma$ in this proof. As in Lemma 7.4, we let $\gamma_{\xi,\eta}$ be the nearest-point projection of e to a bi-infinite geodesic $[\xi, \eta]$ in $(\Gamma, \mathsf{d}_\Gamma)$. By Proposition 7.2, there exists $R > 0$, depending only on r, such that $\eta \in O_R^\Gamma(e, g)$. Write the geodesic ray $[e, \xi]$ as a sequence $\{g_k\}_{k \geq 0}$ with $g_0 = e$. Since $g \in [e, \xi]$ by the hypothesis, we have $g_i = g$ for some $i \geq 0$. Then for $r_R > 0$ given in Lemma 7.4, there exists $j \geq i$ such that

$$\mathsf{d}_\Gamma(\gamma_{\xi,\eta}, g_j) \leq r_R.$$

Let $n_1 \geq 0$ and $D \geq 0$ be given by Lemma 6.4 (uniform progression lemma) and Theorem 4.1 (coarse triangle inequality) respectively. We then have

$$\begin{aligned}\mathsf{d}_\psi(o, g_j o) &\geq \mathsf{d}_\psi(o, g_{i-n_1} o) + 1 \\ &\geq \mathsf{d}_\psi(o, go) - \mathsf{d}_\psi(g_{i-n_1} o, go) - D + 1\end{aligned}.$$

Since $g = g_i$, we have $\mathsf{d}_\psi(g_{i-n_1} o, go) \leq Q_\psi(n_1 + 1)$ where Q_ψ is the constant in Proposition 6.3. Hence we deduce by setting $D' := Q_\psi(n_1 + 1) + D$ that

$$\mathsf{d}_\psi(o, g_j o) \geq \mathsf{d}_\psi(o, go) - D'.$$

On the other hand, applying the coarse triangle inequality (Theorem 4.1) again, we have

$$\mathsf{d}_\psi(o, g_j o) \leq \mathsf{d}_\psi(o, \gamma_{\xi,\eta} o) + \mathsf{d}_\psi(\gamma_{\xi,\eta} o, g_j o) + D.$$

Since $\mathsf{d}_\Gamma(\gamma_{\xi,\eta}, g_j) \leq r_R$, we have $\mathsf{d}_\psi(\gamma_{\xi,\eta} o, g_j o) \leq Q_\psi(r_R + 1)$ by Proposition 6.3, and hence

$$\mathsf{d}_\psi(o, \gamma_{\xi,\eta} o) \geq \mathsf{d}_\psi(o, go) - D' - Q_\psi(r_R + 1) - D.$$

Since we have $|\psi(\mathcal{G}^\theta(\xi, \eta)) - \mathsf{d}_\psi(o, \gamma_{\xi,\eta} o)| < \|\psi\| C_1$ with C_1 given by Lemma 5.4,

$$\psi(\mathcal{G}^\theta(\xi, \eta)) \geq \mathsf{d}_\psi(o, go) - D' - Q_\psi(r_R + 1) - D - \|\psi\| C_1.$$

Setting $c' := e^{D' + Q_\psi(r_R + 1) + D + \|\psi\| C_1}$, we have

$$d_\psi(\xi, \eta) \leq c' e^{-\mathsf{d}_\psi(o, go)}.$$

Hence $\eta \in B_\psi(\xi, c' e^{-\mathsf{d}_\psi(o, go)})$ as desired. $\square$

8. Ahlfors regularity of Patterson-Sullivan measures

As before, let Γ be a θ-Anosov subgroup of a connected semisimple real algebraic group G. Recall from Theorem 3.4 that the space of Γ-Patterson-Sullivan measures on Λ_θ is parameterized by the set

$$\mathscr{T}_\Gamma = \{\psi \in \mathfrak{a}_\theta^* : \psi \text{ is tangent to } \psi_\Gamma^\theta\}.$$

We continue to use the notation ν_ψ for the unique (Γ, ψ)-Patterson-Sullivan measure on Λ_θ. Recall that d_ψ is the premetric on Λ_θ defined by $d_\psi(\xi, \eta) = e^{-\psi(\mathcal{G}(\xi, \eta))}$ for all $\xi \neq \eta$ in Λ_θ and $B_\psi(\xi, r) = \{\eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r\}$.

The Ahlfors regularity is an important notion in fractal geometry:

Definition 8.1. A premetric space (Z, d) is called Ahlfors s-regular if there exist a Borel measure ν on Z and $C \geq 1$ so that for all $z \in Z$ and $r \in [0, \operatorname{diam} Z)$,

$$C^{-1} r^s \leq \nu(B(z, r)) \leq C r^s$$

where $B(z, r) = \{w \in Z : d(z, w) < r\}$. Such a measure ν is also called Ahlfors s-regular.

The goal of this section is to deduce the following from Theorem 6.2:

Theorem 8.2. *For any symmetric $\psi \in \mathscr{T}_\Gamma$, the measure ν_ψ is Ahlfors one-regular on (Λ_θ, d_ψ).*

Remark 8.3. When Γ is a convex cocompact subgroup of $G = \mathrm{SO}^\circ(n, 1)$, $\mathscr{T}_\Gamma$ is a singleton consisting of the critical exponent δ_Γ (more precisely, the multiplication by δ_Γ on $\mathbb{R}$), and the metric d_{δ_Γ} is the δ_Γ-power of a K-invariant Riemannian metric on $\mathbb{S}^{n-1}$. Hence Theorem 8.2 is equivalent to Sullivan's theorem [55, Theorem 7] that the Patterson-Sullivan measure of a Riemannian ball of radius r is comparable to r^{δ_Γ}.

We use the higher rank version of Sullivan's shadow lemma. The following is a special case of [33, Lemma 7.2]:

Lemma 8.4 (Shadow lemma)**.** *Let $\Gamma < G$ be a non-elementary θ-Anosov subgroup. For all large enough $R > 0$, there exists $c_0 = c_0(\psi, R) \geq 1$ such that for all $\gamma \in \Gamma$,*

$$c_0^{-1} e^{-\psi(\mu_\theta(\gamma))} \leq \nu_\psi(O_R^\theta(o, \gamma o)) \leq c_0 e^{-\psi(\mu_\theta(\gamma))}.$$

Proof of Theorem 8.2. By Lemma 3.7 and Remark 5.7, it suffices to consider the case of $\theta = \mathrm{i}(\theta)$. Let c and R_0 be the constants as in Theorem 6.2. Fix $\xi \in \Lambda_\theta$ and $0 < r < \operatorname{diam}(\Lambda_\theta, d_\psi)$. Write the geodesic ray $[e, \xi]$ as $\{\gamma_k\}_{k \geq 0}$ in $(\Gamma, \mathsf{d}_\Gamma)$. Setting

$$i = i_r := \max\{k : r \leq c e^{-\mathsf{d}_\psi(o, \gamma_k o)}\}, \tag{8.1}$$

Theorem 6.2 implies that for any $R > R_0$,

$$B_\psi(\xi, r) \subset B_\psi(\xi, c e^{-\mathsf{d}_\psi(o, \gamma_i o)}) \subset O_R^\theta(o, \gamma_i o).$$

By Lemma 8.4, we get

$$\nu_\psi(B_\psi(\xi, r)) \leq c_0 e^{-\mathsf{d}_\psi(o, \gamma_i o)}. \tag{8.2}$$

By the coarse triangle inequality of d_ψ (Theorem 4.1), we have

$$\mathsf{d}_\psi(o, \gamma_{i+1} o) \leq \mathsf{d}_\psi(o, \gamma_i o) + \mathsf{d}_\psi(\gamma_i o, \gamma_{i+1} o) + D$$

where D is as in loc. cit. Since $\mathsf{d}_\psi(\gamma_i o, \gamma_{i+1} o) \leq 2Q_\psi$ with Q_ψ in Proposition 6.3, we have

$$\mathsf{d}_\psi(o, \gamma_{i+1} o) \leq \mathsf{d}_\psi(o, \gamma_i o) + D'$$

where $D' = D + 2Q_\psi$. This implies

$$c e^{-D'} e^{-\mathsf{d}_\psi(o, \gamma_i o)} \leq c e^{-\mathsf{d}_\psi(o, \gamma_{i+1} o)} < r$$

where the last inequality follows from the definition of $i = i_r$ in (8.1). Hence we deduce from (8.2) that

$$\nu_\psi(B_\psi(\xi, r)) \leq (c_0 e^{D'}/c) \cdot r.$$

Now let $c' = c'_R > 0$ be given by Theorem 6.2 and set

$$j = j_r := \min\{k : c' e^{-\mathsf{d}_\psi(o,\gamma_k o)} \leq r\}. \tag{8.3}$$

By Theorem 6.2, we have

$$O_R^\theta(o, \gamma_j o) \cap \Lambda_\theta \subset B_\psi(\xi, c' e^{-\mathsf{d}_\psi(o,\gamma_j o)}) \subset B_\psi(\xi, r),$$

and hence applying Lemma 8.4 yields

$$c_0^{-1} e^{-\mathsf{d}_\psi(o,\gamma_j o)} \leq \nu_\psi(B_\psi(\xi, r)).$$

By the minimality of $j = j_r$ as defined in (8.3) and the coarse triangle inequality of d_ψ (Theorem 4.1), we have

$$r < c' e^{-\mathsf{d}_\psi(o,\gamma_{j-1} o)} \leq c' e^D e^{-\mathsf{d}_\psi(o,\gamma_j o) + \mathsf{d}_\psi(\gamma_{j-1} o, \gamma_j o)}.$$

Recalling that $\mathsf{d}_\psi(\gamma_{j-1} o, \gamma_j o) \leq 2Q_\psi$ and $D' = D + 2Q_\psi$, we have

$$r < c' e^{D'} e^{-\mathsf{d}_\psi(o,\gamma_j o)}$$

and hence

$$(c_0 c' e^{D'})^{-1} \cdot r \leq \nu_\psi(B_\psi(\xi, r)).$$

Therefore the theorem is proved with $c_1 = \max(c_0 e^{D'} c^{-1}, c_0 c' e^{D'})$. $\square$

9. Hausdorff measures on limit sets

Let $\Gamma < G$ be a θ-Anosov subgroup where G is a connected semisimple real algebraic group. For a linear form $\psi \in \mathfrak{a}_\theta^*$ which is positive on $\mathcal{L} - \{0\}$, consider the associated conformal premetric d_ψ on Λ_θ. For $s > 0$, we denote by $\mathcal{H}_\psi^s$ the associated Hausdorff measure of dimension s, that is, for any subset $B \subset \Lambda_\theta$, let

$$\mathcal{H}_\psi^s(B) := \lim_{\varepsilon \to 0} \inf \left\{ \sum_i (\operatorname{diam}_\psi U_i)^s : B \subset \bigcup_i U_i, \ \sup_i \operatorname{diam}_\psi U_i \leq \varepsilon \right\} \tag{9.1}$$

where $\operatorname{diam}_\psi U = \sup_{\xi,\eta \in U} d_\psi(\xi, \eta)$. This is an outer measure which induces a Borel measure on Λ_θ (see [22], [18, Appendix A]). For $s = 1$, we simply write $\mathcal{H}_\psi$ for $\mathcal{H}_\psi^1$. Recall that $\mathscr{T}_\Gamma$ is the space of all linear forms tangent to the growth indicator ψ_Γ^θ. In this section, we first deduce the following two theorems from Theorem 8.2, which imply Theorems 1.1 and 1.3. We also prove Theorems 9.12 and 1.4.

Theorem 9.1. *For any symmetric $\psi \in \mathscr{T}_\Gamma$, the associated Patterson-Sullivan measure ν_ψ coincides with the one-dimensional Hausdorff measure $\mathcal{H}_\psi$, up to a constant multiple. In other words, $\mathcal{H}_\psi$ is the unique (Γ, ψ)-conformal measure on Λ_θ (up to a constant multiple).*

We also show that the symmetric hypothesis is necessary:

Theorem 9.2. *If $\psi \in \mathscr{T}_\Gamma$ is not symmetric and Γ is Zariski dense, then ν_ψ is not comparable to $\mathcal{H}_\psi^s$ for any $s > 0$.*

Remark 9.3. If $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L} - \{0\}$, then $\delta_\psi \psi \in \mathscr{T}_\Gamma$. Since $\mathcal{H}_{\delta_\psi \psi} = \mathcal{H}_\psi^{\delta_\psi}$, Theorem 9.1 says that if ψ is symmetric in addition,

$$\mathcal{H}_\psi^{\delta_\psi} = \nu_{\delta_\psi \psi} \quad \text{up to a constant multiple.} \tag{9.2}$$

Remark 9.4. For a special class of symmetric ψ whose gradient lies in the interior of $\mathfrak{a}_\theta^+$, Dey-Kapovich [18, Corollary 4.8] showed that $(\Gamma o, \mathsf{d}_\psi)$ is a Gromov hyperbolic space and they proved Theorem 9.1 relying upon the work of Coornaert [16] which gives the positivity and finiteness of $\mathcal{H}_\psi$ for the Gromov hyperbolic space. In our generality, $(\Gamma o, \mathsf{d}_\psi)$ is not even a metric space, and hence their approach cannot be extended.

The main work is to establish the positivity and the finiteness of $\mathcal{H}_\psi$ and the key ingredient is the Ahlfors regular property of ν_ψ obtained in Theorem 8.2. For example, positivity of $\mathcal{H}_\psi$ is a standard consequence of the Ahlfors regularity of (Λ_θ, d_ψ). However, we cannot conclude finiteness of $\mathcal{H}_\psi$ directly from Ahlfors regularity due to the lack of the triangle inequality:

Proposition 9.5 (Positivity)**.** *For any symmetric $\psi \in \mathscr{T}_\Gamma$, we have*

$$\mathcal{H}_\psi(\Lambda_\theta) > 0.$$

Proof. Fix $\varepsilon > 0$ and a countable cover $\{U_i\}_{i \in \mathbb{N}}$ such that $\operatorname{diam}_\psi U_i \le \varepsilon$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we choose $\xi_i \in U_i$. By Theorem 8.2, we have

$$\sum_{i \in \mathbb{N}} \operatorname{diam}_\psi U_i \gg \sum_{i \in \mathbb{N}} \nu_\psi(B_\psi(\xi_i, \operatorname{diam}_\psi U_i))$$

where the implied constant depends only on ψ. Since $\Lambda_\theta \subset \bigcup_{i \in \mathbb{N}} U_i \subset \bigcup_{i \in \mathbb{N}} B_\psi(\xi_i, \operatorname{diam}_\psi U_i)$, it follows that $\sum_{i \in \mathbb{N}} \operatorname{diam}_\psi U_i \gg \nu_\psi(\Lambda_\theta) = 1$. Since $\{U_i\}_{i \in \mathbb{N}}$ is an arbitrary countable cover, it follows that $\mathcal{H}_{\psi, \varepsilon}(\Lambda_\theta) \gg 1$ with the implied constant depending only on ψ. Since $\varepsilon > 0$ is arbitrary, we have $\mathcal{H}_\psi(\Lambda_\theta) > 0$. ☐

Proposition 9.6 (Finiteness)**.** *For any symmetric $\psi \in \mathscr{T}_\Gamma$, we have*

$$\mathcal{H}_\psi(\Lambda_\theta) < \infty.$$

Proof. Let $N = N(\psi)$ and $N_0 = N_0(\psi)$ be the constants given in Proposition 5.3 and Lemma 5.6 respectively. Fix $\varepsilon > 0$. Since Λ_θ is compact, we have a finite cover Λ_θ by $\bigcup_{i=1}^n B_\psi(\xi_i, \frac{\varepsilon}{2NN_0})$ for some finite set $\xi_1, \cdots, \xi_n \in \Lambda_\theta$. Applying the Vitali covering type lemma (Lemma 5.6), there exists a disjoint subcollection $B_\psi(\xi_{i_1}, \frac{\varepsilon}{2NN_0}), \cdots, B_\psi(\xi_{i_k}, \frac{\varepsilon}{2NN_0})$ such that

$$\Lambda_\theta \subset \bigcup_{j=1}^k B_\psi(\xi_{i_j}, \tfrac{\varepsilon}{2N}).$$

Since $\operatorname{diam}_\psi B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N}) \le \varepsilon$ for each $1 \le j \le k$ by Proposition 5.3(2), we have

$$\mathcal{H}_{\psi,\varepsilon}(\Lambda_\theta) \le \sum_{j=1}^k \operatorname{diam}_\psi B_\psi(\xi_{i_j}, \tfrac{\varepsilon}{2N}) \le k \cdot \varepsilon.$$

Applying Theorem 8.2, we obtain

$$k \cdot \varepsilon \ll \sum_{j=1}^k \nu_\psi(B_\psi(\xi_{i_j}, \tfrac{\varepsilon}{2NN_0})) = \nu_\psi \left(\bigcup_{j=1}^k B_\psi(\xi_{i_j}, \tfrac{\varepsilon}{2NN_0}) \right) \le \nu_\psi(\Lambda_\theta) = 1$$

where the equality follows from the disjointness. This implies $\mathcal{H}_{\psi,\varepsilon}(\Lambda_\theta) \ll 1$. Since ε is arbitrary, we have $\mathcal{H}_\psi(\Lambda_\theta) \ll 1$. $\square$

Hence $\mathcal{H}_\psi$ is a *non-trivial* measure on Λ_θ. It is also (Γ, ψ)-conformal:

Lemma 9.7 (Conformality)**.** *For any symmetric $\psi \in \mathscr{T}_\Gamma$, we have*

$$\frac{d\gamma_* \mathcal{H}_\psi}{d\mathcal{H}_\psi}(\xi) = e^{\psi(\beta_\xi^\theta(e,\gamma))}$$

for all $\gamma \in \Gamma$ and $\xi \in \Lambda_\theta$.

Proof. Since d_ψ is invariant under the Γ-equivariant homeomorphism $p : \Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_\theta$ by the definition of d_ψ (Remark 5.7), the measure $(\mathcal{H}_\psi, \Lambda_\theta)$ is the push-forward of the Hausdorff measure $(\mathcal{H}_\psi, \Lambda_{\theta \cup \mathrm{i}(\theta)})$ via p. Therefore it suffices to prove this lemma assuming that $\theta = \mathrm{i}(\theta)$. We simply write $\beta^\theta = \beta$ in this proof to ease the notations.

Fix $\gamma \in \Gamma$ and $\xi \in \Lambda_\theta$. Let $U \subset \Lambda_\theta$ be a small open neighborhood of ξ. To estimate $\gamma_* \mathcal{H}_\psi(U)$ in terms of $\mathcal{H}_\psi(U)$, we fix $\varepsilon > 0$ and take any cover $\{U_i\}_{i \in \mathbb{N}}$ of U such that $\operatorname{diam}_\psi U_i \le \varepsilon$ and that $U \cap U_i \neq \emptyset$ for all $i \in \mathbb{N}$.

For simplicity, we write $s_{\xi,R}(\gamma) := \sup_{\eta \in B_\psi(\xi,R)} e^{\psi(\beta_\eta(e,\gamma))}$. By Lemma 5.2 and Proposition 5.3 with $N = N(\psi) > 0$ therein, we have that for each $i \ge 1$,

$$\operatorname{diam}_\psi \gamma^{-1} U_i \le \sup_{\eta \in U_i} e^{\psi(\beta_\eta(e,\gamma))} \operatorname{diam}_\psi U_i \le s_{\xi,R_\varepsilon}(\gamma) \operatorname{diam}_\psi U_i$$

where $R_\varepsilon = R_\varepsilon(U) = N \cdot (\operatorname{diam}_\psi U + \varepsilon)$. We then have for $\tilde{\varepsilon} := s_{\xi,R_\varepsilon}(\gamma)\varepsilon$,

$$\mathcal{H}_{\psi,\tilde{\varepsilon}}(\gamma^{-1}U) \le \sum_{i \in \mathbb{N}} \operatorname{diam}_\psi \gamma^{-1} U_i \le s_{\xi,R_\varepsilon}(\gamma) \sum_{i \in \mathbb{N}} \operatorname{diam}_\psi U_i.$$

Since $\{U_i\}_{i \in \mathbb{N}}$ is an arbitrary countable open cover of U, the above inequality implies

$$\mathcal{H}_{\psi,\tilde{\varepsilon}}(\gamma^{-1}U) \le s_{\xi,R_\varepsilon}(\gamma) \mathcal{H}_{\psi,\varepsilon}(U).$$

Taking $\varepsilon \to 0$, we have $\tilde{\varepsilon} = s_{\xi,R_\varepsilon}(\gamma)\varepsilon \to 0$ and $R_\varepsilon \to R_U := N \cdot \operatorname{diam}_\psi U$. Therefore

$$\mathcal{H}_\psi(\gamma^{-1}U) \le s_{\xi,R_U}(\gamma) \mathcal{H}_\psi(U). \tag{9.3}$$

Applying (9.3) after replacing U with $\gamma^{-1}U$, and γ by γ^{-1}, we have

$$\mathcal{H}_\psi(U) = \mathcal{H}_\psi(\gamma(\gamma^{-1}U)) \le s_{\gamma^{-1}\xi, R_{\gamma^{-1}U}}(\gamma^{-1})\mathcal{H}_\psi(\gamma^{-1}U). \tag{9.4}$$

If we set $c = \sup_{\zeta\in\Lambda_\theta} e^{\psi(\beta_\zeta(e,\gamma^{-1}))}$, then for any $\eta \in B_\psi(\gamma^{-1}\xi, R_{\gamma^{-1}U})$, it follows from Lemma 5.2 that

$$d_\psi(\xi, \gamma\eta) \le c d_\psi(\gamma^{-1}\xi, \eta) \le cR_{\gamma^{-1}U}.$$

This implies

$$\begin{aligned} s_{\gamma^{-1}\xi, R_{\gamma^{-1}U}}(\gamma^{-1}) &= \sup_{\eta\in B_\psi(\gamma^{-1}\xi, R_{\gamma^{-1}U})} e^{\psi(\beta_\eta(e,\gamma^{-1}))} \\ &\le \sup_{\eta\in B_\psi(\xi, cR_{\gamma^{-1}U})} e^{\psi(\beta_{\gamma^{-1}\eta}(e,\gamma^{-1}))} = \sup_{\eta\in B_\psi(\xi, cR_{\gamma^{-1}U})} e^{\psi(\beta_\eta(\gamma,e))} \end{aligned}$$

Hence we obtain from (9.4) that

$$\mathcal{H}_\psi(U) \le \sup_{\eta\in B_\psi(\xi, cR_{\gamma^{-1}U})} e^{-\psi(\beta_\eta(e,\gamma))}\mathcal{H}_\psi(\gamma^{-1}U).$$

Together with (9.3), we deduce

$$\inf_{\eta\in B_\psi(\xi, cR_{\gamma^{-1}U})} e^{\psi(\beta_\eta(e,\gamma))} \le \frac{\gamma_*\mathcal{H}_\psi(U)}{\mathcal{H}_\psi(U)} \le \sup_{\eta\in B_\psi(\xi, R_U)} e^{\psi(\beta_\eta(e,\gamma))}.$$

Now shrinking $U \to \xi$, we have $R_U, R_{\gamma^{-1}U} \to 0$ and hence the both sides in the above inequality converge to $e^{\psi(\beta_\xi(e,\gamma))}$, by the continuity of the Busemann map $\beta_\eta(e,\gamma)$ on the η-variable. Therefore

$$\frac{d\gamma_*\mathcal{H}_\psi}{d\mathcal{H}_\psi}(\xi) = e^{\psi(\beta_\xi(e,\gamma))}$$

as desired. □

Proof of Theorem 9.1. By Propositions 9.5 and 9.6, we have $\mathcal{H}_\psi(\Lambda_\theta) \in (0,\infty)$. Moreover, it follows from Lemma 9.7 that $\frac{1}{\mathcal{H}_\psi(\Lambda_\theta)}\mathcal{H}_\psi$ is a (Γ,ψ)-Patterson-Sullivan measure. Since there exists a unique (Γ,ψ)-Patterson-Sullivan measure on Λ_θ (Theorem 3.4), this completes the proof. □

Proof of Theorem 9.2. By Lemma 3.7, we may assume $\theta = \mathrm{i}(\theta)$. Since $\psi \ne \psi\circ\mathrm{i}$, two linear forms ψ and $\bar\psi$ are not proportional. Since d_ψ and $d_{\bar\psi}$ are bi-Lipschitz by Proposition 5.5, $\mathcal{H}^s_\psi$ is in the same measure class as $\mathcal{H}^s_{\bar\psi}$ for all $s > 0$. Hence it follows from Theorem 9.1 (see also Remark 9.3) that $\mathcal{H}^s_\psi(\Lambda_\theta) = 0$ or ∞ if $s \ne \delta_{\bar\psi}$. Now it suffices to show that ν_ψ is not comparable to $\mathcal{H}^{\delta_{\bar\psi}}_\psi$. Since ψ and $\bar\psi$ are not proportional, ψ and $\delta_{\bar\psi}\bar\psi$ are two different forms tangent to ψ^θ_Γ. By Theorem 3.4, it follows that ν_ψ is mutually singular to $\nu_{\delta_{\bar\psi}\bar\psi}$. Since the latter is proportional to $\mathcal{H}^{\delta_{\bar\psi}}_{\bar\psi}$ by Theorem 9.1, ν_ψ is singular to $\mathcal{H}^{\delta_{\bar\psi}}_{\bar\psi}$ and hence singular to $\mathcal{H}^{\delta_{\bar\psi}}_\psi$ as well. This finishes the proof. □

Remark 9.8. In fact, without Zariski dense hypothesis, it was shown in [53, Theorem A] that for $\psi_1, \psi_2 \in \mathscr{T}_\Gamma$, ν_{ψ_1} and ν_{ψ_2} are mutually singular unless $\psi_1 = \psi_2$ on $\mathcal{L}_\theta$. Hence Theorem 9.2 holds provided that ψ and $\psi \circ \mathrm{i}$ are not identical on $\mathcal{L}_\theta$.

Critical exponents and Hausdorff dimensions. The Hausdorff dimension of Λ_θ with respect to d_ψ is defined as

$$\dim_\psi \Lambda_\theta := \inf\{s > 0 : \mathcal{H}^s_\psi(\Lambda_\theta) = 0\} = \sup\{s > 0 : \mathcal{H}^s_\psi(\Lambda_\theta) = \infty\}.$$

As a corollary of Theorem 9.1, we obtain the following (Theorem 1.4):

Corollary 9.9. *For any $\psi \in \mathfrak{a}_\theta^*$ positive on $\mathcal{L} - \{0\}$, we have*

$$\delta_{\bar\psi} = \dim_\psi \Lambda_\theta$$

where $\bar\psi = \frac{\psi + \psi\circ \mathrm{i}}{2}$.

Proof. By Proposition 5.5, we have $\dim_\psi \Lambda_\theta = \dim_{\bar\psi} \Lambda_\theta$. Applying Theorem 9.1 to $\delta_{\bar\psi}\bar\psi$ (see Remark 9.3), we have $\mathcal{H}^{\delta_{\bar\psi}}_{\bar\psi}(\Lambda_\theta) \in (0, \infty)$, which implies $\dim_{\bar\psi} \Lambda_\theta = \delta_{\bar\psi}$. This shows the claim. $\square$

For ψ non-symmetric, $\dim_\psi \Lambda_\theta$ is not in general equal to δ_ψ:

Proposition 9.10. *For $\psi \in \mathfrak{a}_\theta^*$ positive on $\mathcal{L} - \{0\}$, we have*

$$\delta_{\bar\psi} \le \delta_\psi.$$

If Γ is Zariski dense, then the equality holds if and only if $\psi = \psi \circ \mathrm{i}$.

Proof. As before, we may assume $\theta = \mathrm{i}(\theta)$. Suppose that $\psi \neq \psi \circ \mathrm{i}$. Note that $\delta_\psi = \delta_{\psi\circ\mathrm{i}}$ and hence both $\delta_\psi\psi$ and $\delta_\psi(\psi\circ\mathrm{i})$ are tangent to the θ-growth indicator ψ_Γ^θ ([31, Theorem 2.5], [33, Lemma 4.5]). We then have $\psi_\Gamma^\theta \le \delta_\psi\psi$ and $\psi_\Gamma^\theta \le \delta_\psi(\psi \circ \mathrm{i})$. Hence $\psi_\Gamma^\theta \le \delta_\psi\bar\psi$. Since $\delta_{\bar\psi}\bar\psi$ is tangent to ψ_Γ^θ, it follows that $\delta_{\bar\psi} \le \delta_\psi$.

Now suppose that Γ is Zariski dense and that $\psi \neq \psi \circ \mathrm{i}$. By Theorem 3.3, there exists a unique unit vector $u = u_{\delta_\psi\psi} \in \operatorname{int}\mathcal{L}_\theta$ such that $\psi_\Gamma^\theta(u) = \delta_\psi\psi(u)$. Since ψ_Γ^θ is i-invariant, it implies that $\psi_\Gamma^\theta(\mathrm{i}(u)) = \delta_\psi(\psi\circ\mathrm{i})(\mathrm{i}(u))$. On the other hand, $u \neq \mathrm{i}(u)$ by Theorem 3.3. Hence the inequality $\psi_\Gamma^\theta \le \delta_\psi\psi$ and $\psi_\Gamma^\theta \le \delta_\psi(\psi \circ \mathrm{i})$ cannot become equalities simultaneously at the same vector. This implies that $\psi_\Gamma^\theta < \delta_\psi\bar\psi$ and hence $\delta_{\bar\psi} < \delta_\psi$. $\square$

By Corollary 9.9 and Proposition 9.10, we obtain:

Corollary 9.11. *Let Γ be Zariski dense in G. For any $\psi \in \mathscr{T}_\Gamma$, we have $\dim_\psi \Lambda_\theta \le 1$ and the equality holds if and only if ψ is symmetric.*

We now prove the Ahlfors regularity of (Λ_θ, d_ψ) for general $\psi \in \mathscr{T}_\Gamma$:

Theorem 9.12. *Let Γ be a θ-Anosov subgroup. For any $\psi \in \mathscr{T}_\Gamma$, the premetric space (Λ_θ, d_ψ) is Ahlfors s-regular for some $0 < s \le 1$. Moreover if Γ is Zariski dense, we have $s = 1$ if and only if ψ is symmetric.*

Proof. Let $\psi \in \mathscr{T}_\Gamma$. By Proposition 5.5, the identity map $(\Lambda_\theta, d_\psi) \to (\Lambda_\theta, d_{\bar\psi})$ is bi-Lipschitz. Noting that $\delta_{\bar\psi}\bar\psi \in \mathscr{T}_\Gamma$ by Lemma 3.6, we denote by $\nu := \nu_{\delta_{\bar\psi}\bar\psi}$ the $(\Gamma, \delta_{\bar\psi}\bar\psi)$-Patterson-Sullivan measure on Λ_θ. By Theorem 8.2, for any $\xi \in \Lambda_\theta$ and $r \in [0, \operatorname{diam}_{\bar\psi} \Lambda_\theta)$, we have

$$\nu(B_{\delta_{\bar\psi}\bar\psi}(\xi, r)) \asymp r. \tag{9.5}$$

Since $B_{\delta_{\bar\psi}\bar\psi}(\xi, r^{\delta_{\bar\psi}}) = B_{\bar\psi}(\xi, r)$ and the identity map $(\Lambda_\theta, d_\psi) \to (\Lambda_\theta, d_{\bar\psi})$ is bi-Lipschitz by Proposition 5.5, (9.5) implies

$$\nu(B_\psi(\xi, r)) \asymp r^{\delta_{\bar\psi}}.$$

Recall $\delta_\psi = 1$ for $\psi \in \mathscr{T}_\Gamma$ (Lemma 3.6). Hence $\delta_{\bar\psi} < 1$ for ψ non-symmetric and Γ Zariski dense by Proposition 9.10. This finishes the proof. ☐

Analyticity of Hausdorff dimensions. For a hyperbolic group Σ, a representation $\sigma : \Sigma \to G$ is called θ-Anosov if σ has a finite kernel and its image $\sigma(\Sigma)$ is a θ-Anosov subgroup of G. For a given $\psi \in \mathfrak{a}_\theta^*$ which is non-negative on $\mathfrak{a}_\theta^+$, the ψ-critical exponents $\delta_\psi(\sigma(\Sigma))$ vary analytically on analytic families of θ-Anosov representations σ in the variety $\operatorname{Hom}(\Sigma, G)$ by Bridgeman-Canary-Labourie-Sambarino [10, Proposition 8.1] (see also [15, Section 4.4]). Hence the following is an immediate consequence of Corollary 9.9:

Corollary 9.13. *Let Σ be a non-elementary*[9] *hyperbolic group and $\psi \in \mathfrak{a}_\theta^*$ be non-negative on $\mathfrak{a}_\theta^+$. Let $\mathcal{D} \subset \operatorname{Hom}(\Sigma, G)$ be an analytic family of θ-Anosov representations. Then*

$$\sigma \mapsto \dim_\psi \Lambda_\theta(\sigma(\Sigma))$$

is analytic on $\mathcal{D}$.

(p, q)-Hausdorff dimensions. Let Σ be a non-elementary convex cocompact subgroup of $\mathrm{SO}^\circ(n, 1) = \mathrm{Isom}^+(\mathbb{H}^n)$, $n \geq 2$. Let $\mathrm{CC}(\Sigma)$ denote the space

$$\{\sigma : \Sigma \to \mathrm{SO}^\circ(n, 1) : \text{convex cocompact, faithful representation}\}/\sim$$

where the equivalence relation is given by conjugations. As in the introduction, for $\sigma \in \mathrm{CC}(\Sigma)$, we denote by $\Lambda_\sigma \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ the limit set of the self-joining subgroup $\Sigma_\sigma := (\mathrm{id} \times \sigma)(\Sigma) < \mathrm{SO}^\circ(n, 1) \times \mathrm{SO}^\circ(n, 1)$, which is well-defined up to translations. The Hausdorff dimension of Λ_σ with respect to a Riemannian metric on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ is equal to $\max\left(\dim \Lambda_\Sigma, \dim \Lambda_{\sigma(\Sigma)}\right)$, where $\Lambda_\Sigma \subset \mathbb{S}^{n-1}$ and $\Lambda_{\sigma(\Sigma)} \subset \mathbb{S}^{n-1}$ are limit sets of Σ and $\sigma(\Sigma)$ respectively and Hausdorff dimensions are computed with respect to a Riemannian metric on $\mathbb{S}^{n-1}$ [30, Theorem 1.1].

For a pair (p, q) of positive real numbers, let $d_{p,q}$ be the premetric on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ defined as

$$d_{p,q}(\xi, \eta) = d_{\mathbb{S}^{n-1}}(\xi_1, \eta_1)^p d_{\mathbb{S}^{n-1}}(\xi_2, \eta_2)^q$$

[9] A hyperbolic group is non-elementary if its Gromov boundary has at least three points.

where $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and $d_{\mathbb{S}^{n-1}}$ is a Riemannian metric on $\mathbb{S}^{n-1}$. We also denote by $\dim_{p,q}$ the Hausdorff dimension with respect to $d_{p,q}$. Let $\delta_{p,q}(\sigma)$ denote the critical exponent of the series

$$s \mapsto \sum_{\gamma \in \Sigma} e^{-s(pd_{\mathbb{H}^n}(o,\gamma o) + qd_{\mathbb{H}^n}(o,\sigma(\gamma)o))}.$$

We deduce the following:

Corollary 9.14. *Let Σ be a non-elementary convex cocompact subgroup of $\mathrm{SO}^\circ(n,1)$, $n \geq 2$. Let p, q be positive real numbers.*

(1) *For any $\sigma \in \mathrm{CC}(\Sigma)$, we have*
$$\dim_{p,q} \Lambda_\sigma = \delta_{p,q}(\sigma).$$
(2) *For any $\sigma \in \mathrm{CC}(\Sigma)$, we have*
$$\dim_{p,q} \Lambda_\sigma \leq \left(\frac{p}{\dim \Lambda_\Sigma} + \frac{q}{\dim \Lambda_{\sigma(\Sigma)}} \right)^{-1}$$
and the equality holds if and only if $\rho = \mathrm{id}$.
(3) *Moreover the map*
$$\sigma \mapsto \dim_{p,q} \Lambda_\sigma$$
is an analytic function on any analytic subfamily of $\mathrm{CC}(\Sigma)$. In particular, for $n = 2, 3$, it is analytic on $\mathrm{CC}(\Sigma)$.

Proof. Identifying the Cartan subspace $\mathfrak{a}$ of $\mathrm{SO}^\circ(n,1) \times \mathrm{SO}^\circ(n,1)$ with $\mathbb{R}^2$ and $\mathfrak{a}^+$ with $\mathbb{R}^2_{\geq 0}$, consider the linear form $\Psi \in \mathfrak{a}^*$ defined by $\Psi(u_1, u_2) = pu_1 + qu_2$. Since $d_{\mathbb{S}^{n-1}}(\xi, \eta) = e^{-\mathcal{G}(\xi,\eta)}$ is a $\mathrm{SO}(n)$-invariant metric and hence a Riemannian metric where $\mathcal{G}$ is the Gromov product on $\mathbb{S}^{n-1} \simeq \partial \mathbb{H}^n$, we have $d_\Psi = d_{p,q}$, where d_Ψ is defined in (5.1). Since the opposition involution i is trivial for $\mathrm{SO}^\circ(n,1) \times \mathrm{SO}^\circ(n,1)$, the linear form Ψ is symmetric and hence the claims (1) and (3) respectively follow from Corollary 9.9 and Corollary 9.13 applied to any analytic subfamily of $\{(\mathrm{id} \times \sigma) : \Sigma \to \mathrm{SO}^\circ(n,1) \times \mathrm{SO}^\circ(n,1) : \sigma \in \mathrm{CC}(\Sigma)\}$. For $n = 2, 3$, $\mathrm{CC}(\Sigma)$ is known to be analytic (cf. [5], [41, Theorem 10.8], [27]). Hence the last claim of (3) follows. Claim (2) follows from (1) and the following Theorem 9.15. $\square$

The following theorem is due to Bishop-Steger [6, Theorem 2] for $n = 2$ and to Burger [13, Theorem 1(a)] in general. We denote by δ_Σ the critical exponent of Σ, the abscissa of convergence of the Poincaré series $s \mapsto \sum_{\gamma \in \Sigma} e^{-sd_{\mathbb{H}^n}(o,\gamma o)}$.

Theorem 9.15. *For each $\sigma \in \mathrm{CC}(\Sigma)$, we have*
$$\delta_{p,q}(\sigma) \leq \left(\frac{p}{\delta_\Sigma} + \frac{q}{\delta_{\sigma(\Sigma)}} \right)^{-1}$$
and the equality holds if and only if $\sigma = \mathrm{id}$.

Proof. We explain how to deduce this from [13, Theorem 1(a)]. We again identify the Cartan subspace $\mathfrak{a}$ of $\mathrm{SO}^\circ(n,1) \times \mathrm{SO}^\circ(n,1)$ with $\mathbb{R}^2$. For each $i = 1, 2$, denote by δ_i the α_i-critical exponent of $\Sigma_\sigma = (\mathrm{id} \times \sigma)(\Sigma)$ where $\alpha_i : \mathfrak{a} \to \mathbb{R}$, $(u_1, u_2) \mapsto u_i$. Then $\delta_1 = \delta_\Sigma$ and $\delta_2 = \delta_{\sigma(\Sigma)}$. If we set $\alpha_i' := \delta_i \alpha_i$, then $\delta_{\alpha_i'} = 1$ for each $i = 1, 2$ and hence Burger's theorem [13, Theorem 1(a)] implies that the critical exponent of any convex combination of α_1' and α_2' is at most one and is equal to one only when $\sigma = \mathrm{id}$.

Since

$$p\alpha_1 + q\alpha_2 = \left(\frac{p}{\delta_1} + \frac{q}{\delta_2}\right) \frac{\frac{p}{\delta_1}\alpha_1' + \frac{q}{\delta_2}\alpha_2'}{\frac{p}{\delta_1} + \frac{q}{\delta_2}} = \left(\frac{p}{\delta_1} + \frac{q}{\delta_2}\right) \Psi_0$$

where $\Psi_0 := \frac{\frac{p}{\delta_1}\alpha_1' + \frac{q}{\delta_2}\alpha_2'}{\frac{p}{\delta_1} + \frac{q}{\delta_2}}$ is a convex combination of α_1' and α_2', we get

$$\delta_{p,q}(\sigma) = \delta_{p\alpha_1 + q\alpha_2} = \left(\frac{p}{\delta_1} + \frac{q}{\delta_2}\right)^{-1} \delta_{\Psi_0} \le \left(\frac{p}{\delta_1} + \frac{q}{\delta_2}\right)^{-1} \tag{9.6}$$

and the equality holds if and only if $\sigma = \mathrm{id}$. □

Remark 9.16. We remark that in Corollary 9.14, the hypothesis $p, q > 0$ was imposed to be able to consider *all* $\sigma \in \mathrm{CC}(\Sigma)$. If we replace $\mathrm{CC}(\Sigma)$ by a subset $\mathcal{D} \subset \mathrm{CC}(\Sigma)$, then Corollary 9.14 holds for any $p, q \in \mathbb{R}$ such that $pu_1 + qu_2 > 0$ for all non-zero $(u_1, u_2) \in \mathcal{L}(\Sigma_\sigma)$ and all $\sigma \in \mathcal{D}$.

10. Hausdorff dimensions with respect to Riemannian metrics

Let G be a connected semisimple real algebraic group. As before, let θ be a non-empty subset of the set Π of simple roots of $(\mathfrak{g}, \mathfrak{a})$. We denote by

$$\dim \Lambda_\theta$$

the Hausdorff dimension of Λ_θ with respect to a Riemannian metric d_{Riem} on $\mathcal{F}_\theta$. As any two Riemannian metrics are bi-Lipschitz, $\dim \Lambda_\theta$ is well-defined independent of the choice of a Riemannian metric. In this section, we present an estimate on $\dim \Lambda_\theta$.

Tits representations and the sum of Tits weights. Let $\mathbf{G}$ be the semisimple algebraic group defined over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})^\circ$. There exists an exact sequence $\tilde{\mathbf{G}} \to_{\tilde{p}} \mathbf{G} \to_{\bar{p}} \bar{\mathbf{G}}$ where $\tilde{\mathbf{G}}$ and $\bar{\mathbf{G}}$ are respectively simply connected and adjoint semisimple $\mathbb{R}$-groups and $\tilde{p}$ and $\bar{p}$ are central $\mathbb{R}$-isogenies ([8], [42, Proposition 1.4.11]).

Recall that for $\alpha \in \Pi$, ω_α denotes the (restricted) fundamental weight associated to α as defined in (2.1). The first part of the following theorem immediately follows as a special case of a theorem of Tits [58], and the second part is remarked in [3] and proved in [54].

Theorem 10.1 ([58, Theorem 7.2], [54, Lemma 2.13])**.** *For each $\alpha \in \Pi$, there exists an irreducible $\mathbb{R}$-representation $\tilde{\rho}_\alpha : \tilde{\mathbf{G}} \to \mathrm{GL}(\mathbf{V}_\alpha)$ whose highest (restricted) weight χ_α is equal to $k_\alpha \omega_\alpha$ for some positive integer k_α and whose highest weight space is one-dimensional. Moreover, all weights of $\tilde{\rho}_\alpha$ are χ_α,*

$\chi_\alpha - \alpha$ and weights of the form $\chi_\alpha - \alpha - \sum_{\beta\in\Pi} n_\beta \beta$ with n_β non-negative integers.

For each $\alpha \in \Pi$, we fix once and for all a representation $\tilde{\rho}_\alpha : \tilde{\mathbf{G}} \to \mathrm{GL}(\mathbf{V}_\alpha)$ as in Theorem 10.1 with *minimal* k_α. Since $\tilde{p}$ and $\bar{p}$ are central isogenies and $\tilde{p}(\tilde{\mathbf{G}}(\mathbb{R})) = G$, the representation $\tilde{\rho}_\alpha$ induces a projective representation

$$\rho_\alpha : G \to \mathrm{PGL}(V_\alpha) \tag{10.1}$$

where $V_\alpha = \mathbf{V}_\alpha(\mathbb{R})$. Since the restriction of $\tilde{\rho}_\alpha$ to $\tilde{\mathbf{G}}(\mathbb{R})$ and ρ_α induce the same representation of the Lie algebra $\mathfrak{g}$ to $\mathfrak{gl}(V_\alpha)$, their restricted weights are the same. We call

$$\rho_\alpha \quad \text{and} \quad \chi_\alpha \tag{10.2}$$

the Tits representation and the Tits weight associated to α respectively.

Let ρ denote the half-sum of all positive roots for $(\mathfrak{g}, \mathfrak{a})$ counted with multiplicity: $2\rho = \sum_{\alpha\in\Phi^+} (\dim \mathfrak{g}^\alpha)\alpha$. In terms of the restricted fundamental weights ω_α, we then have

$$\rho = \sum_{\alpha\in\Pi} c_\alpha \omega_\alpha \tag{10.3}$$

where $c_\alpha = \dim \mathfrak{g}^\alpha$ if 2α is not a root, and $c_\alpha = \frac{1}{2}(\dim \mathfrak{g}^\alpha + 2\dim \mathfrak{g}^{2\alpha})$ otherwise (cf. [9]). If G is split over $\mathbb{R}$, we have $\chi_\alpha = \omega_\alpha$ for all $\alpha \in \Pi$ and hence $\sum_{\alpha\in\Pi} \chi_\alpha = \rho$. In general, we do not have this identity, which motivates the following definition:

Definition 10.2. Define c_G to be the infimum number $c > 0$ such that

$$\sum_{\alpha\in\Pi} \chi_\alpha \le c \cdot \rho.$$

By our choice of the Tits representation of G, note that c_G depends only on the Lie algebra $\mathfrak{g}$; hence we sometimes write $\mathsf{c}_G = c_{\mathfrak{g}}$. The proof of the following lemma was provided by I. Smilga:

Lemma 10.3. *We have $\sum_{\alpha\in\Pi} \chi_\alpha \le \rho$, and hence*

$$\mathsf{c}_{\mathfrak{g}} \le 1.$$

Proof. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ containing $\mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{h}$, we have a natural restriction map $\pi : \mathfrak{h}^* \to \mathfrak{a}^*$. Recall the restricted fundamental weights $\omega_1, \cdots, \omega_s$ defined in (2.1) where $s = \dim \mathfrak{a}$; they form a basis of $\mathfrak{a}^*$. We denote by $\bar{\omega}_1, \cdots, \bar{\omega}_r$ the fundamental weights of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ where $r = \dim \mathfrak{h}$, which were chosen compatibly with ω_j's so that π sends each $\bar{\omega}_j$ to some linear combination $\sum_i c_{j,i}\omega_i$ where $c_{j,i}$ are non-negative integers. They form a basis of $\mathfrak{h}^*$.

Set $\rho_{\mathbb{C}} = \sum_{i=1}^r \bar{\omega}_i$, which is equal to the half-sum of all positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$. Then

$$\rho = \pi(\rho_{\mathbb{C}}) = \sum_i d_i \omega_i$$

where $d_i = \sum_j c_{j,i} \in \mathbb{N}$. For the Tits weights $\chi_i = \kappa_i \omega_i$ for $i = 1, \cdots, s$, recall that κ_i is the smallest positive integer κ such that $\kappa\omega_i$ is proximal, that is, its highest weight space is one-dimensional. In view of the Killing form, we may consider $\mathfrak{a}^*$ as a subset of $\mathfrak{h}^*$. We have the following facts:

- a representation with the highest weight $\chi \in \mathfrak{h}^*$ is proximal if and only if χ actually lies in $\mathfrak{a}^*$ [1, Theorem 6.3];
- each coefficient κ_i is either 1 or 2 [4, Section 2.3].

We now claim that $\kappa_i \leq d_i$ for all $1 \leq i \leq s$; this implies that $\sum_i \chi_i = \sum_i \kappa_i \omega_i \leq \rho$. This is clear if $\kappa_i = 1$, since $d_i \geq 1$. So suppose that $\kappa_i = 2$ and let us show that $d_i \geq 2$. Then $\chi_i = 2\omega_i$ lies in $\mathfrak{a}^*$ and is an integral weight; hence it is equal to some linear combination $\sum_j c_j \bar{\omega}_j$ with non-negative integer coefficients c_j. Moreover the sum $\sum_j c_j$ cannot exceed 2, because π has to map $\sum_j c_j \bar{\omega}_j$ to $2\omega_i$, and it maps each $\bar{\omega}_j$ to some non-zero sum of the ω_k's. So we are left with three cases:

(a) $2\omega_i = \bar{\omega}_j$ for some j;
(b) $2\omega_i = \bar{\omega}_j + \bar{\omega}_k$ for some distinct j and k;
(c) $2\omega_i = 2\bar{\omega}_j$ for some j.

We can rule out case (c), because then $\omega_i = \bar{\omega}_j$ would be proximal which contradicts $\kappa_i = 2$. In case (a), we get that $c_{j,i} = 2$ and hence d_i is at least 2. In case (b), applying π on both sides, we necessarily have $\pi(\bar{\omega}_j) = \pi(\bar{\omega}_k) = \omega_i$, so $c_{j,i} = c_{k,i} = 1$ and hence d_i is also at least 2. ☐

The bound on c_G can be improved in certain cases. For example, for $\mathfrak{g} = \mathfrak{so}(n,1)$, $n \geq 2$, we have $\Pi = \{\alpha\}$, $\rho = \frac{n-1}{2}\alpha$ and $\chi_\alpha = \omega_\alpha = \frac{\alpha}{2}$; hence $\mathsf{c}_{\mathfrak{g}} = \frac{1}{n-1}$.

Riemannian metric on $\mathcal{F}_\theta$. For each $\alpha \in \Pi$, we denote by V_α^+ the highest weight space of ρ_α and by $V_\alpha^<$ its unique complementary A-invariant subspace in V_α. Then the map $g \in G \mapsto (\rho_\alpha(g) V_\alpha^+)_{\alpha\in\theta}$ factors through a proper immersion

$$\mathcal{F}_\theta \to \prod_{\alpha\in\theta} \mathbb{P}(V_\alpha). \tag{10.4}$$

Let $\langle \cdot, \cdot \rangle_\alpha$ be a K-invariant inner product on V_α with respect to which A is symmetric, so that V_α^+ is perpendicular to $V_\alpha^<$. We denote by $\|\cdot\|_\alpha$ the norm on V_α induced by $\langle \cdot, \cdot \rangle_\alpha$. We also use the notation $\|\cdot\|_\alpha$ for a bi-$\rho_\alpha(K)$-invariant norm on $\mathrm{GL}(V_\alpha)$. The angle $\angle(E, F)$ between a line E and a subspace F is defined as minimum of all angles between all non-zero $v \in E$ and non-zero $w \in F$.

We write $gV_\alpha^+ := \rho_\alpha(g) V_\alpha^+$ and $gV_\alpha^< := \rho_\alpha(g) V_\alpha^<$ for $g \in G$ and $\alpha \in \Pi$. Up to a Lipschitz equivalence, the Riemannian distance d_{Riem} on $\mathcal{F}_\theta = G/P_\theta$ satisfies that for all $g_1, g_2 \in G$,

$$d_{\mathrm{Riem}}(g_1 P_\theta, g_2 P_\theta) = \sqrt{\sum_{\alpha\in\theta} \sin^2 \angle(g_1 V_\alpha^+, g_2 V_\alpha^+)}.$$

The Gromov product $\mathcal{G}$ on $\mathcal{F}^{(2)}$ can be expressed in terms of angles between appropriate subspaces as follows:

Lemma 10.4 ([49, Lemma 6.4], [37, Lemma 3.11])**.** *For $(\xi, \eta) \in \mathcal{F}^{(2)}$, we have that for any $\alpha \in \Pi$,*

$$2\chi_\alpha(\mathcal{G}(\xi, \eta)) = -\log \sin \angle(gV_\alpha^+, gV_\alpha^<)$$

where $g \in G$ is such that $\xi = gP$ and $\eta = gw_0P$.

We then have the following estimates on the Riemannian distance using Gromov products and Tits weights:

Lemma 10.5. *There exists a constant $C > 0$ such that for all $g \in G$,*

$$d_{\mathrm{Riem}}(gP_\theta, gw_0P_\theta) \geq C \left(\sum_{\alpha \in \theta} e^{-4\chi_\alpha(\mathcal{G}(gP, gw_0P))} \right)^{1/2}.$$

Proof. We first note that for each $\alpha \in \Pi$, $w_0V_\alpha^+ \subset V_\alpha^<$; to see this, recall that $V_\alpha^<$ is the sum of all weight subspaces of V_α whose weight is not equal to χ_α. On the other hand, $w_0V_\alpha^+$ is a weight space with the weight given by $\chi_\alpha \circ \mathrm{Ad}_{w_0} = -\chi_\alpha \circ \mathrm{i}$. Since $-\chi_\alpha \circ \mathrm{i}(\mathfrak{a}^+) \leq 0$ while $\chi_\alpha(\mathfrak{a}^+) \geq 0$, we have $\chi \circ \mathrm{Ad}_{w_0} \neq \chi_\alpha$, which shows $w_0V_\alpha^+ \subset V_\alpha^<$.

Therefore for all $g \in G$,

$$\sin^2 \angle(gV_\alpha^+, gV_\alpha^<) \leq \sin^2 \angle(gV_\alpha^+, gw_0V_\alpha^+)$$

Hence, up to a Lipschitz constant, we have that for all $g \in G$,

$$\begin{aligned} d_{\mathrm{Riem}}(gP_\theta, gw_0P_\theta) &= \sqrt{\sum_{\alpha \in \theta} \sin^2 \angle(gV_\alpha^+, gw_0V_\alpha^+)} \\ &\geq \sqrt{\sum_{\alpha \in \theta} \sin^2 \angle(gV_\alpha^+, gV_\alpha^<)} \\ &= \left(\sum_{\alpha \in \theta} e^{-4\chi_\alpha(\mathcal{G}(gP, gw_0P))} \right)^{1/2} \end{aligned}$$

where the last equality follows from Lemma 10.4. ☐

Lower bounds. In the rest of this section, we assume that

$$\Gamma \text{ is a } \theta\text{-Anosov subgroup of } G.$$

Since the Tits weights $\{\chi_\alpha : \alpha \in \theta\}$ form a basis of $\mathfrak{a}_\theta^*$, each linear form $\psi \in \mathfrak{a}_\theta^*$ can be uniquely written as $\psi = \sum_{\alpha \in \theta} \kappa_{\psi,\alpha}\chi_\alpha$ with $\kappa_{\psi,\alpha} \in \mathbb{R}$. We consider the following height of ψ:

$$\kappa_\psi := \sum_{\alpha \in \theta} \kappa_{\psi,\alpha} \in \mathbb{R}$$

Denote by E_θ the collection of all linear forms which are non-negative linear combinations of $\{\chi_\alpha : \alpha \in \theta\}$. That is,

$$\mathsf{E}_\theta := \{\psi \in \mathfrak{a}_\theta^* : \kappa_{\psi,\alpha} \geq 0 \text{ for all } \alpha \in \theta\}. \tag{10.5}$$

Since $\chi_\alpha > 0$ on $\operatorname{int}\mathfrak{a}_\theta^+$ for all $\alpha \in \theta$, each non-zero $\psi \in \mathsf{E}_\theta$ is positive on $\operatorname{int}\mathfrak{a}_\theta^+$. Since $\mathcal{L}_\theta - \{0\} \subset \operatorname{int}\mathfrak{a}_\theta^+$ by Theorem 3.8(2), each non-zero $\psi \in \mathsf{E}_\theta$ is positive on $\mathcal{L}_\theta - \{0\}$ and hence we have the corresponding conformal premetric d_ψ on Λ_θ discussed in section 5:

Lemma 10.6. *For any non-zero $\psi \in \mathsf{E}_\theta$, the identity map*

$$(\Lambda_\theta, d_{\mathrm{Riem}}) \to (\Lambda_\theta, d_\psi^{2/\kappa_\psi})$$

is a Lipschitz map.

Proof. For each $\xi \neq \eta \in \Lambda_\theta$, there exists $g \in G$ such that $\xi = gP_\theta$ and $\eta = gw_0P_\theta$ (Theorem 3.8(4)). By Lemma 10.5, we have that for each $\alpha \in \theta$, up to a Lipschitz constant,

$$d_{\mathrm{Riem}}(\xi, \eta) \geq \left(\sum_{\alpha\in\theta} e^{-4\chi_\alpha(\mathcal{G}(gP, gw_0P))} \right)^{1/2} \geq e^{-2\chi_\alpha(\mathcal{G}(gP, gw_0P))}. \tag{10.6}$$

Recalling

$$d_{2\chi_\alpha}(\xi, \eta) = e^{-2\chi_\alpha(\mathcal{G}(gP, gw_0P))}$$

and writing $\psi = \sum_{\alpha\in\theta} \kappa_{\psi,\alpha}\chi_\alpha \in \mathsf{E}_\theta$, since all $\kappa_{\psi,\alpha}$ are non-negative, (10.6) implies

$$d_\psi(\xi, \eta) = \prod_{\alpha\in\theta} d_{2\chi_\alpha}(\xi, \eta)^{\frac{\kappa_{\psi,\alpha}}{2}} \leq \prod_{\alpha\in\theta} d_{\mathrm{Riem}}(\xi, \eta)^{\frac{\kappa_{\psi,\alpha}}{2}} = d_{\mathrm{Riem}}(\xi, \eta)^{\frac{\kappa_\psi}{2}}$$

up to a Lipschitz constant. Hence the claim follows. ☐

Remark 10.7. Since d_ψ and $d_{\bar\psi}$ are bi-Lipschitz (Proposition 5.5), Proposition 6.8 and the above lemma imply that there exist $c, R > 0$ such that for any $\xi \in \Lambda_\theta$ and $g \in [e, \xi]$ in Γ, the shadow $O_R^\theta(o, go) \cap \Lambda_\theta$ contains the Riemannian ball of center ξ and of radius $ce^{-\frac{2}{\kappa_\psi}\mathsf{d}_\psi(o,go)}$.

Theorem 10.8. *For any non-zero $\psi \in \mathsf{E}_\theta$, we have*

$$\dim \Lambda_\theta \geq \frac{\kappa_\psi}{2} \cdot \dim_\psi \Lambda_\theta = \frac{\kappa_\psi}{2} \cdot \delta_{\bar\psi}.$$

Proof. It follows from Lemma 10.6 and a standard property of Hausdorff dimension that we get

$$\dim \Lambda_\theta \geq \frac{\kappa_\psi}{2} \cdot \dim_\psi \Lambda_\theta.$$

Since $\dim_\psi \Lambda_\theta = \delta_{\bar\psi}$ by Corollary 9.9, the claim follows. ☐

Applying Theorem 10.8 to each χ_α, $\alpha \in \theta$, we obtain the following uniform lower bound on the Hausdorff dimension of all non-elementary θ-Anosov subgroups:

Corollary 10.9. *We have*

$$\dim \Lambda_\theta \geq \max_{\alpha\in\theta} \delta_{\chi_\alpha + \chi_{\mathrm{i}(\alpha)}}.$$

Example 10.10. For $G = \mathrm{PSL}_n(\mathbb{R})$, we have $\Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}$ where

$$\alpha_i : \mathrm{diag}(a_1, \cdots, a_n) \mapsto a_i - a_{i+1}.$$

Let $1 \le p \le n-1$. Since χ_{α_p} is equal to the fundamental weight ω_p which is given by $\omega_p(\mathrm{diag}(a_1, \cdots, a_n)) = a_1 + \cdots + a_p$, we deduce from Corollary 10.9 that for all non-elementary α_p-Anosov subgroups of $\mathrm{PSL}_n(\mathbb{R})$, we have

$$\dim \Lambda_{\alpha_p} \ge \delta_{\omega_p + \omega_{n-p}}.$$

When $p = 1$, this lower bound is obtained in [18, Theorem 10.1].

The following upper bound in Proposition 10.11 was obtained in ([47, Theorem B], [14, Theorem 1.2]) for $G = \mathrm{PSL}_n(\mathbb{R})$ and θ is a singleton.

Proposition 10.11. *We have*

$$\dim \Lambda_\theta \le \max_{\alpha\in\theta} \delta_\alpha.$$

Proof. Via the proper immersion of $\mathcal{F}_\theta$ into $\prod_{\alpha\in\theta} \mathbb{P}(V_\alpha)$ as discussed in (10.4), we may consider the following metric on $\mathcal{F}_\theta$: for $g_1, g_2 \in G$,

$$d_{\mathcal{F}_\theta}(g_1 P_\theta, g_2 P_\theta) = \max_{\alpha\in\theta} d_{\mathbb{P}(V_\alpha)}(g_1 V_\alpha^+, g_2 V_\alpha^+)$$

where $d_{\mathbb{P}(V_\alpha)}$ is the metric on $\mathbb{P}(V_\alpha)$ given by $d_{\mathbb{P}(V_\alpha)}(v_1, v_2) = \sin\angle(v_1, v_2)$. Then $d_{\mathcal{F}_\theta}$ is Lipschitz equivalent to the Riemannian metric on $\mathcal{F}_\theta$ and hence we can use $d_{\mathcal{F}_\theta}$ to compute $\dim \Lambda_\theta$.

Fix $\alpha \in \theta$ and consider the Tits representation (ρ_α, V_α). We write $V_\alpha = \mathbb{R}^{n_\alpha}$ and $\mathrm{PGL}(V_\alpha) = \mathrm{PGL}_{n_\alpha}(\mathbb{R})$ by fixing a basis. We denote by $\beta_{1,\alpha}$ the simple root of $\mathrm{PGL}_{n_\alpha}(\mathbb{R})$ given by $\beta_{1,\alpha}(\mathrm{diag}(u_1, \cdots, u_{n_\alpha})) = u_1 - u_2$. Since the highest weight of ρ_α is χ_α and the second highest weight is $\chi_\alpha - \alpha$ by Theorem 10.1, we have that for all $\gamma \in \Gamma$,

$$\beta_{1,\alpha}(\mu(\rho_\alpha(\gamma))) = \alpha(\mu(\gamma)). \tag{10.7}$$

Since Γ is an $\{\alpha\}$-Anosov subgroup of G, there exists $C > 1$ such that for all $\gamma \in \Gamma$, $\alpha(\mu(\gamma)) \ge C^{-1}|\gamma| - C$, and hence $\beta_{1,\alpha}(\mu(\rho_\alpha(\gamma))) \ge C^{-1}|\gamma| - C$. Therefore $\rho_\alpha(\Gamma)$ is a $\{\beta_{1,\rho}\}$-Anosov subgroup of $\mathrm{PGL}_{n_\alpha}(\mathbb{R})$.

We denote by $f_\alpha : \partial\Gamma \to \mathbb{P}(V_\alpha)$ the $\rho_\alpha(\Gamma)$-equivariant embedding obtained as the extension of the orbit map of $\rho_\alpha(\Gamma)$ (Theorem 3.8(4)). It is shown in [47, Proposition 3.5, Proposition 3.8] that there exists a constant $C_\alpha > 0$ such that for each $\gamma \in \Gamma$, there exists a ball $\mathcal{B}_\alpha(\gamma)$ of radius $C_\alpha e^{-\beta_{1,\rho}(\mu(\rho_\alpha(\gamma)))}$ in $\mathbb{P}(V_\alpha)$ so that for any $x \in \partial\Gamma$ such that $\gamma \in [e, x]$ in Γ, we have $f_\alpha(x) \in \mathcal{B}_\alpha(\gamma)$. In particular, for every $k \ge 1$, the collection

$$\{\mathcal{B}_\alpha(\gamma) : \gamma \in \Gamma,\ |\gamma| = k\}$$

covers the limit set of $\rho_\alpha(\Gamma)$ in $\mathbb{P}(V_\alpha)$. Hence Λ_θ is covered by the collection

$$\left\{ \prod_{\alpha\in\theta} \mathcal{B}_\alpha(\gamma) : \gamma \in \Gamma, |\gamma| = k \right\}$$

via the immersion $\mathcal{F}_\theta \to \prod_{\alpha\in\theta} \mathbb{P}(V_\alpha)$. Since $\prod_{\alpha\in\theta} \mathcal{B}_\alpha(\gamma)$ has $d_{\mathcal{F}_\theta}$-diameter at most

$$\max_{\alpha\in\theta} C_\alpha e^{-\beta_{1,\rho}(\mu(\rho_\alpha(\gamma)))} \le Ce^{-\min_{\alpha\in\theta}\alpha(\mu(\gamma))}$$

where $C = \max_{\alpha\in\theta} C_\alpha$ by (10.7), we have that for each $s > 0$, the s-dimensional Hausdorff measure $\mathcal{H}^s(\Lambda_\theta)$ with respect to $d_{\mathcal{F}_\theta}$ satisfies

$$\mathcal{H}^s(\Lambda_\theta) \le \limsup_{k\to\infty} C^s \sum_{\gamma\in\Gamma, |\gamma|=k} e^{-s\min_{\alpha\in\theta}\alpha(\mu(\gamma))}.$$

Therefore, denoting by $\delta_{\min_{\alpha\in\theta}\alpha}$ the abscissa of convergence of the series $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\min_{\alpha\in\theta}\alpha(\mu(\gamma))}$, if $s > \delta_{\min_{\alpha\in\theta}\alpha}$, we have $\mathcal{H}^s(\Lambda_\theta) = 0$ and hence

$$\dim \Lambda_\theta \le \delta_{\min_{\alpha\in\theta}\alpha}.$$

On the other hand, we have

$$\frac{1}{\#\theta} \sum_{\alpha\in\theta}\sum_{\gamma\in\Gamma} e^{-s\alpha(\mu(\gamma))} \le \sum_{\gamma\in\Gamma} e^{-s\min_{\alpha\in\theta}\alpha(\mu(\gamma))} \le \sum_{\alpha\in\theta}\sum_{\gamma\in\Gamma} e^{-s\alpha(\mu(\gamma))}.$$

The first inequality implies $\max_{\alpha\in\theta}\delta_\alpha \le \delta_{\min_{\alpha\in\theta}\alpha}$ and the second gives $\delta_{\min_{\alpha\in\theta}\alpha} \le \max_{\alpha\in\theta}\delta_\alpha$. Hence $\delta_{\min_{\alpha\in\theta}\alpha} = \max_{\alpha\in\theta}\delta_\alpha$, which completes the proof. ☐

Theorem 1.7 is a combination of Corollary 10.9 and Proposition 10.11.

11. Growth indicator bounds and applications to the L^2-spectrum

As before, let $\Gamma < G$ be a θ-Anosov subgroup where G is a connected semisimple real algebraic group. In this final section, we deduce bounds on the growth indicator $\psi_\Gamma^\theta : \mathfrak{a}_\theta \to [0,\infty)\cup\{-\infty\}$ of Γ (see Definition 3.1) from Corollary 10.9. Recall Tits weights χ_α, $\alpha \in \Pi$, of G from (10.2). We have the following (Corollary 1.8):

Corollary 11.1. *We have*

$$\psi_\Gamma^{\theta\cup\mathrm{i}(\theta)} \le \dim \Lambda_\theta \cdot \min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}). \tag{11.1}$$

Moreover,

$$\psi_\Gamma \le \dim \Lambda_\theta \cdot \min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}). \tag{11.2}$$

Proof. For any linear form $\psi \in \mathfrak{a}^*_{\theta\cup\mathrm{i}(\theta)}$ positive on $\mathcal{L}_{\theta\cup\mathrm{i}(\theta)} - \{0\}$, the scaled linear from $\delta_\psi\psi$ is tangent to the growth indicator (Lemma 3.6). Hence it follows from Corollary 10.9 that for each $\alpha \in \theta$, we have

$$\psi_\Gamma^{\theta\cup\mathrm{i}(\theta)} \le \delta_{\chi_\alpha+\chi_{\mathrm{i}(\alpha)}} \cdot (\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \le \dim \Lambda_\theta \cdot (\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \quad \text{on } \mathfrak{a}_{\theta\cup\mathrm{i}(\theta)}.$$

Therefore taking minimum among $\alpha \in \theta$ finishes the proof of (11.1).

By [33, Lemma 3.12], we have

$$\psi_\Gamma \le \psi_\Gamma^{\theta\cup\mathrm{i}(\theta)} \circ p_{\theta\cup\mathrm{i}(\theta)} \quad \text{on } \mathfrak{a}.$$

Hence by (11.1), we have

$$\psi_\Gamma \le \dim \Lambda_\theta \cdot \min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \circ p_{\theta\cup \mathrm{i}(\theta)}.$$

Since the linear form $\chi_\alpha + \chi_{\mathrm{i}(\alpha)} \in \mathfrak{a}^*_{\theta\cup\mathrm{i}(\theta)}$ is $p_{\theta\cup\mathrm{i}(\theta)}$-invariant for each $\alpha\in\theta$, (11.2) follows. $\square$

Using the constant c_G in Definition 10.2, we have

$$\min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \le \frac{1}{\#\theta}\sum_{\alpha\in\theta}\chi_\alpha + \chi_{\mathrm{i}(\alpha)} \le \frac{2}{\#\theta}\sum_{\alpha\in\Pi}\chi_\alpha \le \frac{2\mathsf{c}_G}{\#\theta}\cdot\rho \quad \text{on } \mathfrak{a}^+.$$

Moreover, if $\theta\cap\mathrm{i}(\theta)=\emptyset$, we can sharpen the inequality to

$$\min_{\alpha\in\theta}(\chi_\alpha + \chi_{\mathrm{i}(\alpha)}) \le \frac{1}{\#\theta}\sum_{\alpha\in\theta}\chi_\alpha + \chi_{\mathrm{i}(\alpha)} \le \frac{1}{\#\theta}\sum_{\alpha\in\Pi}\chi_\alpha \le \frac{\mathsf{c}_G}{\#\theta}\cdot\rho \quad \text{on } \mathfrak{a}^+.$$

Hence Corollary 11.1 implies the following:

Corollary 11.2. *For any θ-Anosov subgroup of G, we have*

$$\psi_\Gamma \le \frac{2\mathsf{c}_G \dim\Lambda_\theta}{\#\theta}\cdot\rho. \tag{11.3}$$

Moreover, if $\theta\cap\mathrm{i}(\theta)=\emptyset$, we have

$$\psi_\Gamma \le \frac{\mathsf{c}_G \dim\Lambda_\theta}{\#\theta}\rho.$$

Define the real number $\lambda_0(\Gamma\backslash X)\in[0,\infty)$ as follows:

$$\lambda_0(\Gamma\backslash X) := \inf\left\{\frac{\int_{\Gamma\backslash X}\|\mathrm{grad}\, f\|^2\, d\,\mathrm{Vol}}{\int_{\Gamma\backslash X}|f|^2\, d\,\mathrm{Vol}} \;:\; f\in C_c^\infty(\Gamma\backslash X),\ f\ne 0\right\}. \tag{11.4}$$

This number is equal to the bottom of the L^2-spectrum of $\Gamma\backslash X$ of the Laplace-Beltrami operator [56, Theorem 2.2]. The following was proved in [21, Theorem 1.6] for Π-Anosov subgroups and in [39, Corollary 3] in general:

Theorem 11.3. *If $\Gamma < G$ is a torsion-free discrete subgroup of G with $\psi_\Gamma\le\rho$, then $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0(\Gamma\backslash X)=\|\rho\|^2$.*

Applying Theorem 11.3, we obtain from (11.3) the following criterion on the temperedness of $L^2(\Gamma\backslash G)$ in terms of the size of the limit set (Corollary 1.9):

Corollary 11.4. *Let Γ be a torsion-free θ-Anosov subgroup. Suppose that*

- $\dim\Lambda_\theta \le \frac{\#\theta}{\mathsf{c}_G}$ *if $\theta\cap\mathrm{i}(\theta)=\emptyset$;*
- $\dim\Lambda_\theta \le \frac{\#\theta}{2\mathsf{c}_G}$ *otherwise.*

Then $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0(\Gamma\backslash X)=\|\rho\|^2$.

Moreover λ_0 is not an L^2-eigenvalue ([21], [20], see also [59, Corollary 5.2] for the absence of any principal joint L^2-eigenvalues as well).

Remark 11.5. Indeed, it is shown in [39, Theorem 11] that if $\psi_\Gamma \leq (2-\frac{2}{p})\rho$ for some $p \geq 1$, then $L^2(\Gamma\backslash G)$ is strongly $L^{p+\varepsilon}$-integrable for all $\varepsilon > 0$, that is, for a dense subset of vectors, the associated matrix coefficients belong to $L^{p+\varepsilon}(G)$. Hence if $\dim \Lambda_\theta \leq (2-\frac{2}{p})\frac{\#\theta}{\mathsf{c}_G}$ when $\theta \cap \mathrm{i}(\theta) = \emptyset$ or $\dim \Lambda_\theta \leq (1-\frac{1}{p})\frac{\#\theta}{\mathsf{c}_G}$ in general, we obtain that $L^2(\Gamma\backslash G)$ is strongly $L^{p+\varepsilon}$-integrable for all $\varepsilon > 0$.

Remark 11.6. Using that $\mathsf{c}_G = \frac{1}{n-1}$ for $G = \mathrm{SO}^\circ(n,1)$, Corollary 11.4 says that for a Zariski dense convex cocompact $\Gamma < \mathrm{SO}^\circ(n,1)$, if $\dim \Lambda \leq \frac{n-1}{2}$, then $L^2(\Gamma\backslash \mathrm{SO}^\circ(n,1))$ is tempered and $\lambda_0(\Gamma\backslash\mathbb{H}^n) = \frac{(n-1)^2}{4}$, as shown by Sullivan [56, Theorem 2.21].

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