Mustafin Varieties

Dustin Cartwright, Mathias Häbich, Bernd Sturmfels and Annette Werner

University of California, Berkeley

Abstract

A Mustafin variety is a degeneration of projective space induced by a point configuration in a Bruhat-Tits building. The special fiber is reduced and Cohen-Macaulay, and its irreducible components form interesting combinatorial patterns. For configurations that lie in one apartment, these patterns are regular mixed subdivisions of scaled simplices, and the Mustafin variety is a twisted Veronese variety built from such a subdivision. This connects our study to tropical and toric geometry. For general configurations, the irreducible components of the special fiber are rational varieties, and any blow-up of projective space along a linear subspace arrangement can arise. A detailed study of Mustafin varieties is undertaken for configurations in the Bruhat-Tits tree of $PGL(2)$ and in the two-dimensional building of $PGL(3)$. The latter yields the classification of Mustafin triangles into 38 combinatorial types.

1. Degenerations of projective space

K : field

v : discrete valuation $K^* \to \mathbb{Z}$

R : ring of integers in K

k : residue field of R

V : vector space of dimension $d \geq 2$

$\mathbb{P}(V) = \mathrm{Proj}\,\mathrm{Sym}\,V^*$: projective space of lines in V

L : lattice, free R-module in V of rank d

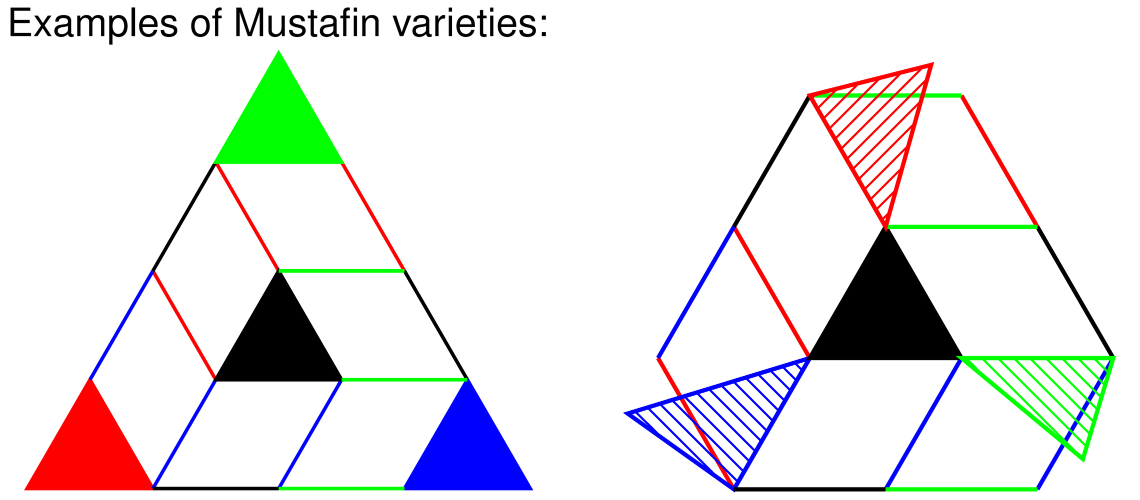
$\mathbb{P}(L) = \mathrm{Proj}\,\mathrm{Sym}\,L^*$: projective space over R

Definition 1. Let $\Gamma = \{L_1, \dots, L_n\}$ be a set of lattices in V. The open immersions $\mathbb{P}(V) \hookrightarrow \mathbb{P}(L_i)$ give rise to a map

$$\mathbb{P}(V) \longrightarrow \mathbb{P}(L_1) \times_R \dots \times_R \mathbb{P}(L_n).$$

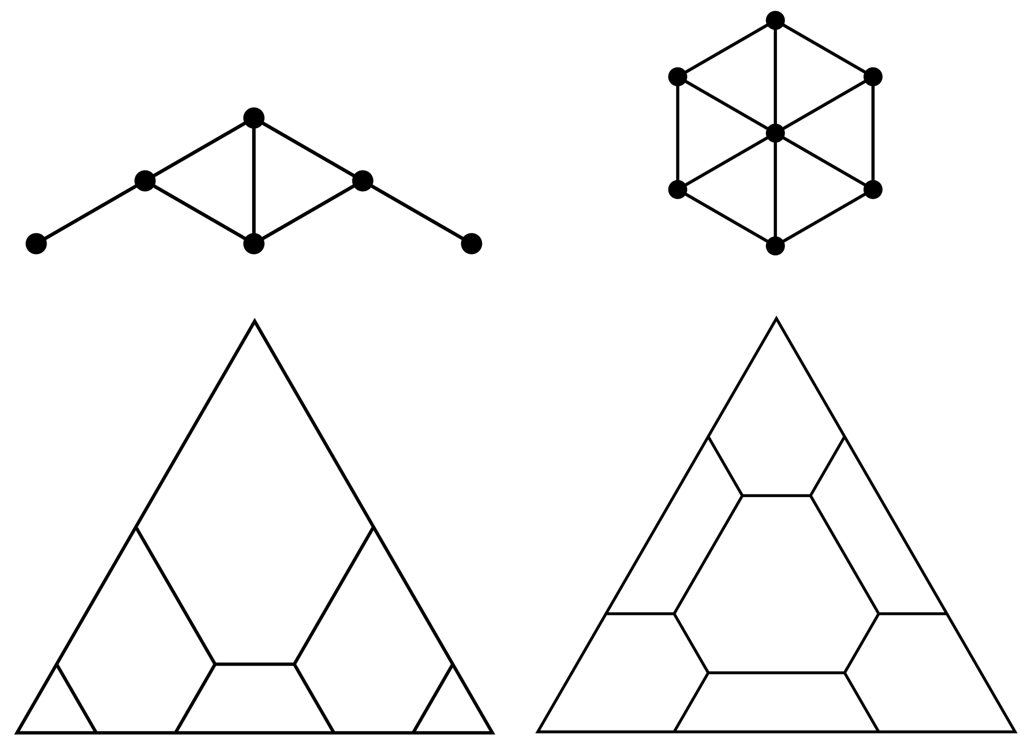
Let $\mathcal{M}(\Gamma)$ be the closure of the image endowed with the reduced scheme structure. We call $\mathcal{M}(\Gamma)$ the **Mustafin variety** associated to the set of lattices Γ. Note that $\mathcal{M}(\Gamma)$ is a scheme over R whose generic fiber is $\mathbb{P}(V)$.

Examples of Mustafin varieties:



The Mustafin variety $\mathcal{M}(\Gamma)$ depends only on the homothety classes of the lattices ($L_i \sim \alpha L_i$ for all $\alpha \in K^\times$). The **Bruhat-Tits building** $\mathfrak{B}_d$ is an infinite simplicial complex whose vertices are homothety classes of lattices.

Theorem 2. *The Mustafin variety $\mathcal{M}(\Gamma)$ is an integral, normal, Cohen-Macaulay scheme which is flat and projective over R. Its special fiber $\mathcal{M}(\Gamma)_k$ is reduced, Cohen-Macaulay and connected. All irreducible components of $\mathcal{M}(\Gamma)_k$ are rational varieties, and their number is at most $\binom{n+d-2}{d-1}$, where $n = |\Gamma|$.*



Theorem 3. *If Γ is a convex subset consisting of n lattice points in the building $\mathfrak{B}_d$, then the Mustafin variety $\mathcal{M}(\Gamma)$ is regular, and its special fiber $\mathcal{M}(\Gamma)_k$ consists of n smooth irreducible components that intersect transversely.*

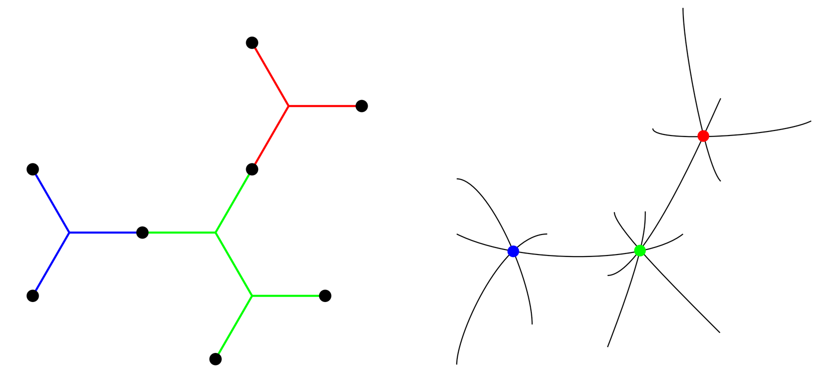
2. Trees ($d = 2$)

For $d = 2$, $\mathfrak{B}_d$ is an infinite tree. Any finite set of vertices Γ is contained in a unique smallest tree T_Γ, which is finite.

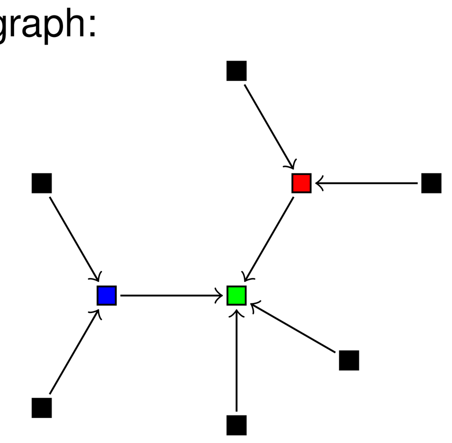
The irreducible components of a 1-dimensional scheme are the vertices of its **reduction complex** and the points of intersection are its edges.

Theorem 4. *The maximal simplices of the reduction complex of $\mathcal{M}(\Gamma)$ correspond to the connected components of the punctured tree $T_\Gamma \backslash \Gamma$. The vertices in each maximal cell are the elements of Γ in the closure of the corresponding component.*

A configuration $\Gamma \subset T_\Gamma$ and its special fiber $\mathcal{M}(\Gamma)_k$:



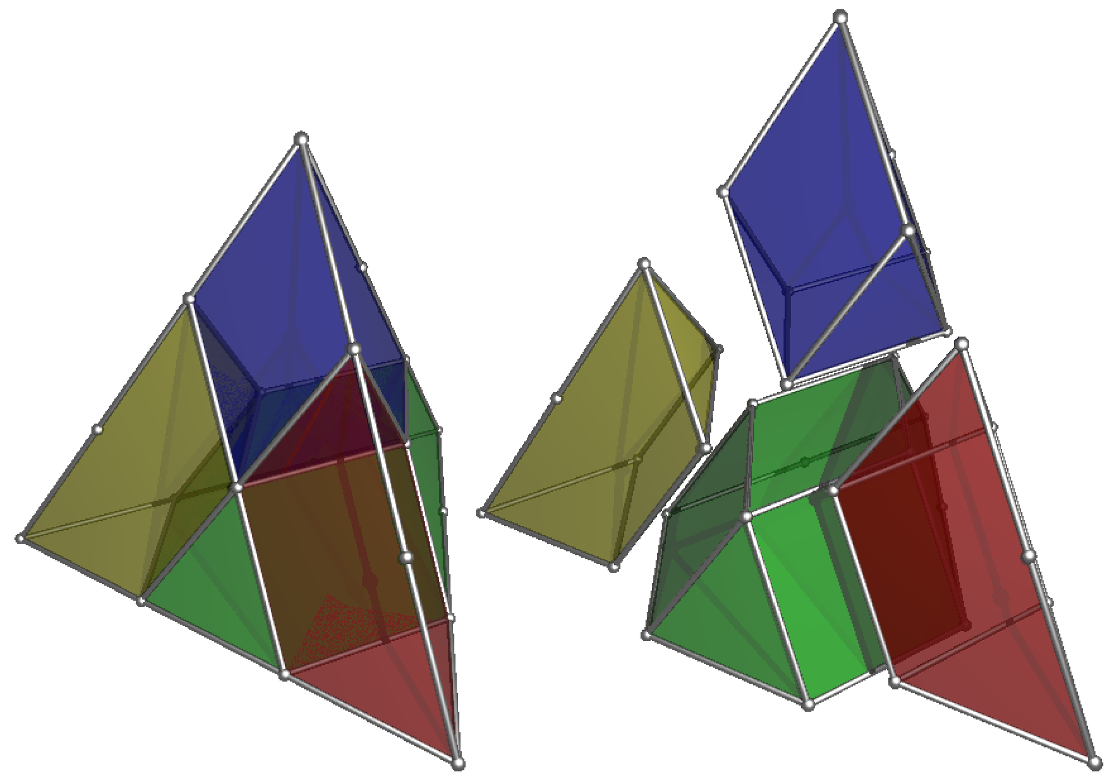
The corresponding special fiber consists of eight projective lines. The special fiber is defined by a monomial, represented by the following directed graph:



3. Points in one apartment

An **apartment** consists of all lattices of the form $\pi^{m_1}Re_1 + \dots + \pi^{m_d}Re_d$, for $e_1, \dots, e_d$ a basis of V. Any pair of lattices are contained in a single apartment.

Theorem 5. *If a configuration Γ is contained in a single apartment, then it defines a tropical polynomial P_Γ. The Mustafin variety $\mathcal{M}(\Gamma)$ is isomorphic to the twisted nth Veronese embedding of the projective space $\mathbb{P}_R^{d-1}$ determined by the tropical polynomial P_Γ. In particular, the special fiber $\mathcal{M}(\Gamma)_k$ equals the union of projective toric varieties corresponding to the cells in the regular mixed subdivision Δ_Γ of the simplex $n \cdot \Delta_{d-1}$.*



The green cell represents a primary component of $\mathcal{M}(\Gamma)_k$ that is a singular toric 3-fold.

4. Components of the special fiber

Definition 6. An irreducible component of $\mathcal{M}(\Gamma)_k$ mapping birationally to the special fiber of the factor $\mathbb{P}(L_i)$ for some $[L_i] \in \Gamma$ is called a **primary component**. All other components of the special fiber are called **secondary components**.

Definition 7. Let $W_1, \dots, W_m$ be linear subspaces in $\mathbb{P}_k^{d-1}$. Let $X_0 = \mathbb{P}_k^{d-1}$ and inductively define X_i to be the blowup of X_{i-1} at the preimage of W_i under $X_{i-1} \to \mathbb{P}_k^{d-1}$. We say that X is a *blow-up of $\mathbb{P}_k^{d-1}$ at a collection of linear subspaces* if X is isomorphic to the variety X_m obtained by this sequence of blow-ups.

Theorem 8. *A projective variety X arises as a primary component of the special fiber $\mathcal{M}(\Gamma)_k$ for some configuration Γ of n lattice points in the Bruhat-Tits building $\mathfrak{B}_d$ if and only if X is the blow-up of $\mathbb{P}_k^{d-1}$ at a collection of $n-1$ linear subspaces.*

The configuration of linear spaces can be described in terms of the configuration Γ. Fix an index i and let C be the primary component of $\mathcal{M}(\Gamma)_k$ corresponding to the lattice class $[L_i]$. For any other point $[L_j]$ in Γ we choose the unique representative L_j such that $L_j \supset \pi L_i$ but $L_j \not\supset L_i$. Then the image of $L_j \cap L_i$ in the quotient $L_i/\pi L_i$ is a proper, non-trivial k-vector subspace, and we denote by W_j the corresponding linear subspace in $\mathbb{P}(L_i)_k$. The component C is the blow-up of $\mathbb{P}(L_i)_k$ at the linear subspaces W_j for all $j \neq i$.

5. Mustafin triangles

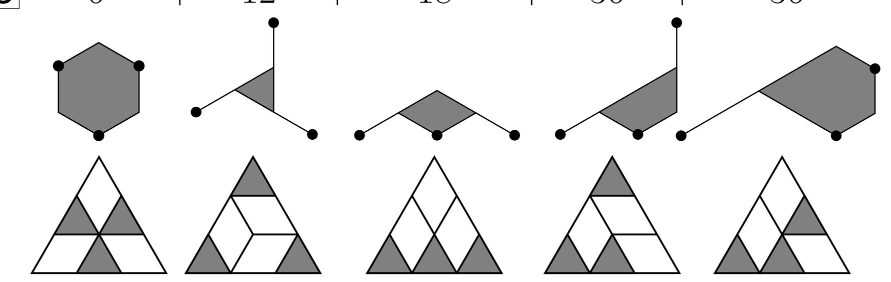
Theorem 9. *There are precisely 38 combinatorial types of Mustafin triangles. In addition to the 18 planar types (contained in one apartment) and* **20** *non-planar types.*

	Number of bent lines			
Number of components	0	1	2	3
3	2+**0**	1+**0**		
4		3+**3**	1+**0**	1+**1**
5			5+**6**	0+**2**
6				5+**8**

The *18* planar types are depicted as follows:

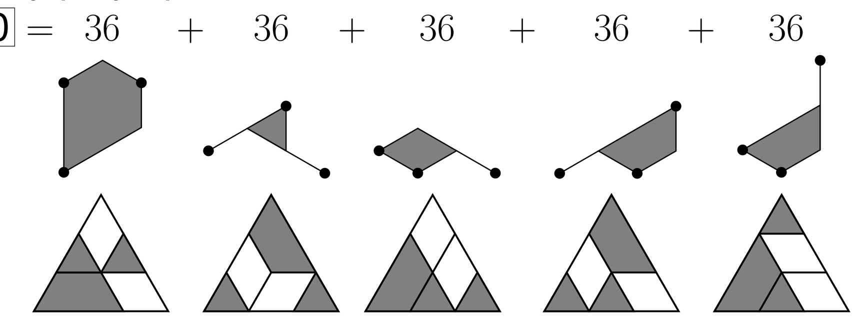
1. *6 components, 3 bent lines, corresponding to vertices of the secondary polytope:*

   108 = 6 + 12 + 18 + 36 + 36

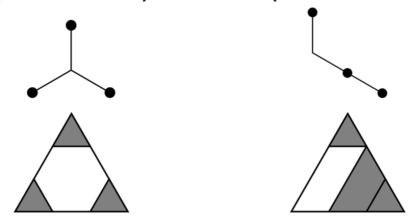
2. *5 components, 2 bent lines, corresponding to edges of the secondary polytope:*

   180 = 36 + 36 + 36 + 36 + 36

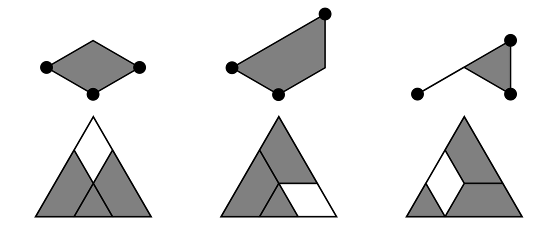
3. *4 components, 3 or 2 bent lines, corresponding to edges of the secondary polytope:*

   6 (3 bends) 36 (2 bends)

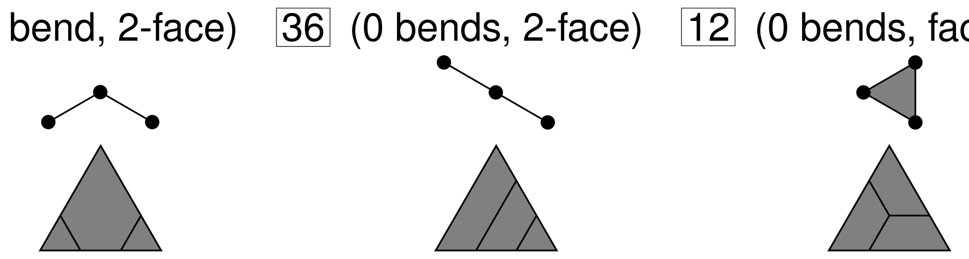
4. *4 components, 1 bent line, corresponding to 2-faces of the secondary polytope:*

   90 = 18 + 36 + 36

5. *3 components, 1 or 0 bent lines, corresponding to 2-faces or facets:*

   18 (1 bend, 2-face) 36 (0 bends, 2-face) 12 (0 bends, facet)

The two combinatorial types of Mustafin triangles with three bent lines and two secondary components are known as paper airplanes: