The Hilbert Scheme of the Diagonal in a Product of Projective Spaces

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Multigraded Hilbert Schemes

Consider a polynomial ring $S = K[z_1, \ldots, z_m]$ with a grading by an Abelian group A. For any function $h \colon A \to \mathbb{N}$, there exists a quasi-projective scheme Hilb_S^h which parametrizes A-homogeneous ideals $I \subset S$ where S/I has Hilbert function h.

This is the *multigraded Hilbert scheme*. [Haiman-Sturmfels 2004]

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Examples:

- $A = \mathbb{Z}$ with the standard grading and suitable h:
  Grothendieck's Hilbert scheme
- $A = \{0\}$: Hilbert scheme of $h(0)$ points in affine m-space
- Any A and $h = 0, 1$: the toric Hilbert scheme

Grading by Column Degree

Let $X = (x_{ij})$ be a $d \times n$-matrix of unknowns. Fix the polynomial ring $K[X]$ with $\mathbb{Z}^n$-grading by column degree, i.e. $\deg(x_{ij}) = e_j$.

The Hilbert function of the polynomial ring $K[X]$ equals

$$\mathbb{N}^n \to \mathbb{N}\,,\ (u_1, \ldots, u_n) \mapsto \prod_{i=1}^{n} \binom{u_i + d - 1}{d - 1}.$$

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Geometry: points on $H_{d,n}$ represent degenerations of the diagonal in a product of projective spaces $(\mathbb{P}^{d-1})^n = \mathbb{P}^{d-1} \times \cdots \times \mathbb{P}^{d-1}$.

Conca's Conjecture

Using an idea suggested to us by Michael Brion, we proved

Theorem (conjectured by Aldo Conca)

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For any ideal in I, we can perform a generic change of coordinates in each column and take the initial ideal.

Key idea: There exists a unique monomial ideal $Z \in H_{d,n}$ which is Borel-fixed in a multigraded sense.

The Borel-fixed Ideal

For $u \in \mathbb{N}^n$, let Z_u be the ideal generated by all unknowns x_{ij} with $1 \leq j \leq n$ and $i \leq u_j$. This is a Borel-fixed prime monomial ideal. The unique Borel-fixed ideal Z on $H_{d,n}$ is the radical ideal

$$Z \ := \ \bigcap_{u \in U} Z_u.$$

$$U = \{(u_1, \ldots, u_n) \in \mathbb{N}^n : u_i \leq d{-}1 \text{ and } \textstyle\sum_i u_i = (n{-}1)(d{-}1)\}.$$

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Proposition

The simplicial complex with Stanley-Reisner ideal Z is shellable.

Corollary

Every ideal I in $H_{d,n}$ is Cohen-Macaulay.

Group Completions

The group $G^n = \mathrm{PGL}(d)^n$ acts on $H_{d,n}$ by transforming each column independently. The stabilizer of $I_2(X)$ is the diagonal subgroup $G \cong \{(A, A, \ldots, A)\}$ of G^n. Thus, the orbit of $I_2(X)$ is the homogeneous space G^n/G, and we write $\overline{G^n/G}$ for its closure.

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In the case of $n = 2$, $H_{d,2}$ is smooth and equals $\overline{G^2/G}$ and coincides with the classical *space of complete collineations*. Our representation as a multigraded Hilbert scheme gives explicit polynomial equations.

Yet Another Space of Trees

Here we restrict to $d = 2$. The points of $H_{2,n}$ are degenerations of the diagonal $\mathbb{P}^1 \to (\mathbb{P}^1)^n$.

Theorem

The multigraded Hilbert scheme $H_{2,n}$ is irreducible, so it equals the compactification $\overline{G^n/G}$. In other words, every $\mathbb{Z}^n$-homogeneous ideal with Hilbert function h is a flat limit of $I_2(X)$.

However, $H_{2,n}$ is singular for $n \geq 4$.

Monomial Ideals in Space of Trees

Theorem

There are $2^n(n+1)^{n-2}$ monomial ideals in $H_{2,n}$, indexed by trees on $n+1$ unlabeled vertices with n labeled, directed edges.

Example: The Hilbert scheme $H_{2,3}$ has 32 monomial ideals, corresponding to the 8 orientations on the claw tree and to the 8 orientations on each of the 3 labeled bivalent trees.

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We construct a graph of the monomial ideals. For any ideal I such that the set of initial ideals of I consists of exactly two monomial ideals, we draw an edge between those monomial ideals.

Theorem

For monomial ideals in $H_{2,n}$*, two monomial ideals are connected by an edge iff the monomial ideals differ by either of the operations:*

1. *Move any subset of the trees attached at a vertex to an adjacent vertex.*
2. *Swap two edges that meet at a bivalent vertex.*

Three Projective Planes

The smallest reducible case is $d = n = 3$, which concerns degenerations of the diagonal plane $\mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

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Theorem

The multigraded Hilbert scheme $H_{3,3}$ is the reduced union of seven irreducible components, each containing a dense $\mathrm{PGL}(3)^3$ orbit:

- *The 16-dimensional main component $\overline{G^3/G}$ is singular.*
- *Three 14-dimensional smooth components are permuted under the S_3-action. A generic point is a reduced union of the blow-up of $\mathbb{P}^2$ at a point, two copies of $\mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1$.*
- *Three 13-dimensional smooth components are permuted under the S_3-action. A generic point is a reduced union of three copies of $\mathbb{P}^2$ and $\mathbb{P}^2$ blown up at three points.*

Poset of Monomial Ideals

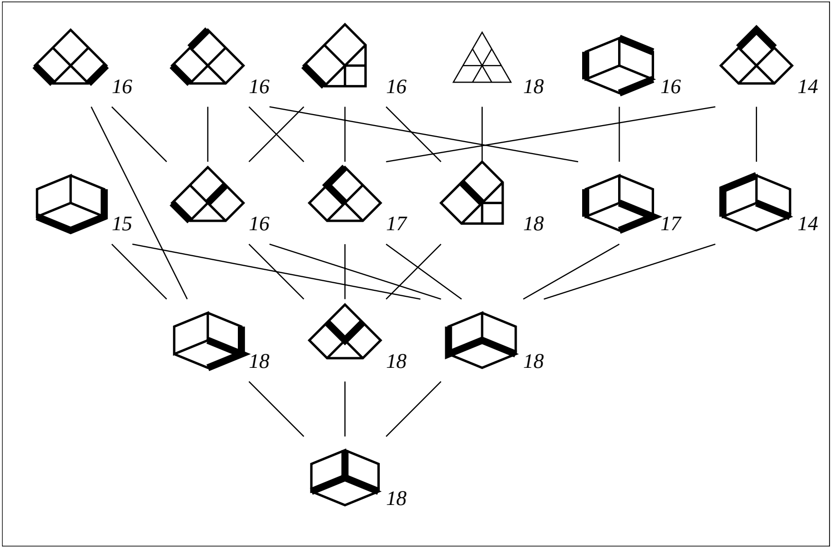

Figure: Partial ordering of the monomial ideals on $H_{3,3}$