The Hilbert Scheme of the Diagonal in a Product of Projective Spaces

Dustin Cartwright
UC Berkeley

joint with Bernd Sturmfels
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Multigraded Hilbert Schemes

Consider a polynomial ring $S = K[z_1, \ldots, z_m]$ with a grading by an Abelian group $A$. For any function $h: A \to \mathbb{N}$, there exists a quasi-projective scheme $\text{Hilb}_S^h$ which parametrizes $A$-homogeneous ideals $I \subset S$ where $S/I$ has Hilbert function $h$.

This is the multigraded Hilbert scheme. [Haiman-Sturmfels 2004]
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Examples:

- $A = \mathbb{Z}$ with the standard grading and suitable $h$: Grothendieck’s Hilbert scheme
- $A = \{0\}$: Hilbert scheme of $h(0)$ points in affine $m$-space
- Any $A$ and $h = 0, 1$: the toric Hilbert scheme
Grading by Column Degree

Let $X = (x_{ij})$ be a $d \times n$-matrix of unknowns. Fix the polynomial ring $K[X]$ with $\mathbb{Z}^n$-grading by column degree, i.e. $\deg(x_{ij}) = e_j$.

The Hilbert function of the polynomial ring $K[X]$ equals

$$\mathbb{N}^n \to \mathbb{N}, \ (u_1, \ldots, u_n) \mapsto \prod_{i=1}^n \binom{u_i + d - 1}{d - 1}.$$
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The ideal of $2 \times 2$-minors $I_2(X)$ has the Hilbert function

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This talk concerns the multigraded Hilbert scheme $H_{d,n} = \text{Hilb}^h_S$. 
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**Geometry:** points on $H_{d,n}$ represent degenerations of the diagonal in a product of projective spaces $(\mathbb{P}^{d-1})^n = \mathbb{P}^{d-1} \times \cdots \times \mathbb{P}^{d-1}$. 
Conca’s Conjecture

Using an idea suggested to us by Michael Brion, we proved

**Theorem (conjectured by Aldo Conca)**

All $\mathbb{Z}^n$-homogeneous ideals $I \subset K[X]$ with multigraded Hilbert function $h$ are radical.
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**Theorem (conjectured by Aldo Conca)**

*All $\mathbb{Z}^n$-homogeneous ideals $I \subset K[X]$ with multigraded Hilbert function $h$ are radical.*

For any ideal in $I$, we can perform a generic change of coordinates in each column and take the initial ideal.

**Key idea:** There exists a unique monomial ideal $Z \in H_{d,n}$ which is Borel-fixed in a multigraded sense.
The Borel-fixed Ideal

For $u \in \mathbb{N}^n$, let $Z_u$ be the ideal generated by all unknowns $x_{ij}$ with $1 \leq j \leq n$ and $i \leq u_j$. This is a Borel-fixed prime monomial ideal. The unique Borel-fixed ideal $Z$ on $H_{d,n}$ is the radical ideal

$$Z := \bigcap_{u \in U} Z_u.$$  

$U = \{(u_1, \ldots, u_n) \in \mathbb{N}^n : u_i \leq d-1 \text{ and } \sum_i u_i = (n-1)(d-1)\}.$
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Proposition

The simplicial complex with Stanley-Reisner ideal $Z$ is shellable.

Corollary

Every ideal $I$ in $H_{d,n}$ is Cohen-Macaulay.
Group Completions

The group $G^n = \text{PGL}(d)^n$ acts on $H_{d,n}$ by transforming each column independently. The stabilizer of $I_2(X)$ is the diagonal subgroup $G \cong \{(A, A, \ldots, A)\}$ of $G^n$. Thus, the orbit of $I_2(X)$ is the homogeneous space $G^n/G$, and we write $\overline{G^n/G}$ for its closure.
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**Theorem**

The equivariant compactification $G^n/G$ is an irreducible component of the multigraded Hilbert scheme $H_{d,n}$. Its dimension is $(d^2 - 1)(n - 1)$. 
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**Theorem**

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In the case of $n = 2$, $H_{d,2}$ is smooth and equals $\overline{G^2/G}$ and coincides with the classical *space of complete collineations*. Our representation as a multigraded Hilbert scheme gives explicit polynomial equations.
Here we restrict to $d = 2$. The points of $H_{2,n}$ are degenerations of the diagonal $\mathbb{P}^1 \to (\mathbb{P}^1)^n$.

**Theorem**

The multigraded Hilbert scheme $H_{2,n}$ is irreducible, so it equals the compactification $G^n/G$. In other words, every $\mathbb{Z}^n$-homogeneous ideal with Hilbert function $h$ is a flat limit of $I_2(X)$.

However, $H_{2,n}$ is singular for $n \geq 4$. 
Monomial Ideals in Space of Trees

Theorem

There are $2^n(n+1)^{n-2}$ monomial ideals in $H_{2,n}$, indexed by trees on $n+1$ unlabeled vertices with $n$ labeled, directed edges.

Example: The Hilbert scheme $H_{2,3}$ has 32 monomial ideals, corresponding to the 8 orientations on the claw tree and to the 8 orientations on each of the 3 labeled bivalent trees.
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We construct a graph of the monomial ideals. For any ideal \(I\) such that the set of initial ideals of \(I\) consists of exactly two monomial ideals, we draw an edge between those monomial ideals.

Theorem
For monomial ideals in \(H_{2,n}\), two monomial ideals are connected by an edge iff the monomial ideals differ by either of the operations:

1. Move any subset of the trees attached at a vertex to an adjacent vertex.
2. Swap two edges that meet at a bivalent vertex.
Three Projective Planes

The smallest reducible case is $d = n = 3$, which concerns degenerations of the diagonal plane $\mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. 
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Theorem

The multigraded Hilbert scheme $H_{3,3}$ is the reduced union of seven irreducible components, each containing a dense $\text{PGL}(3)^3$ orbit:

- The 16-dimensional main component $G^3/G$ is singular.
- Three 14-dimensional smooth components are permuted under the $S_3$-action. A generic point is a reduced union of the blow-up of $\mathbb{P}^2$ at a point, two copies of $\mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1$.
- Three 13-dimensional smooth components are permuted under the $S_3$-action. A generic point is a reduced union of three copies of $\mathbb{P}^2$ and $\mathbb{P}^2$ blown up at three points.
Figure: Partial ordering of the monomial ideals on $H_{3,3}$