Math 54 worksheet, September 21, 2009

1. Find the inverse of
$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix},$$
using both row reduction and the adjugate. Check that you get the same answer.

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

2. Use the adjugate to find a formula for the inverse of a 2×2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3. Compute the determinant of the matrix
$$B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -2 \\ -3 & 1 & 4 \end{bmatrix}.$$
Note that this is just a rearrangement of the rows from Problem 1. In general, if B is a rearrangement of the rows of A, how can you figure out $\det(B)$ in terms of $\det(A)$?

4. Find an example of 3×3 matrices A and B where $\det(A + B)$ does not equal $\det(A) + \det(B)$.

   One example that works is:
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
   and then $\det(A) = 0$, $\det(B) = 0$ and $\det(A + B) \neq 0$.

$$\det B = \det A = 2$$

   In general $\det B = \pm \det A$. The sign is positive if you can get from A to B using an even number of swaps and negative otherwise.

5. In this problem, we will use the properties of the determinant to show that $\text{adj}(A)$ gives a formula for the inverse of a matrix. Let A be an $n \times n$ matrix and let $\text{adj}(A)$ be the adjugate. We will prove that $A \cdot \text{adj}(A) = \det(A) I_n$.

(a) Explain why the product of the ith row of A (a $1 \times n$ matrix) with the ith column of $\mathrm{adj}(A)$ (an $n \times 1$ matrix) is the 1×1 matrix $\det(A)$.

(b) Let $A(i, j)$ be the matrix formed by removing the jth row and replacing it with the ith row. If i and j are different, explain why the product of the ith row of A with the jth column of $\mathrm{adj}(A)$ is $\det(A(i, j))$.

(c) Explain why $\det(A(i, j)) = 0$ if i and j are different.

(d) Explain how this proves the formula that $A \cdot \mathrm{adj}(A) = \det(A) I_n$ and if A is invertible, that $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$.