Math 54 quiz solutions October 12, 2009

1 *Define the* null space *of a matrix (3 points).*

If A is an $n \times m$ matrix, the null space of A is the set of vectors x in $\mathbb{R}^m$ such that $Ax = 0$.

2 *Define what it means for a set of vectors to be* linearly dependent *(3 points).*

A set of vectors $v_1, \ldots, v_n$ is linearly dependent if there exists real numbers $c_1, \ldots, c_n$, not all zero, such that $c_1 v_1 + \cdots c_n v_n = 0$.

3 *Define the* dimension *of a vector space (3 points).*

A basis for a vector space V is a set of vectors which span V and are linearly independent. The dimension of a vector space is the size of a basis.

4 *Give two equivalent definitions of the* rank *of a matrix (3 points). Neither definitions should involve echelon form.*

Let A be an $n \times m$ matrix. Then the columns of A are vectors in $\mathbb{R}^n$ and the rows are vectors in $\mathbb{R}^m$. The rank of A is the dimension of the span of the columns of A, which is equal to the dimension of the span of the rows of A.

5 *Define a* linear transformation *from a vector space V to a vector space W (3 points).*

A linear transformation from V to W is a function T which takes elements in V to elements such that:

1. For any u and v in V, $T(v + w) = T(v) + T(w)$.
2. For any v in V and c a scalar, $T(cv) = cT(v)$.

6 *Recall that the standard basis for $\mathbb{R}^n$ consists of the vectors $e_1, \ldots, e_n$, where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, and so on. Explicitly, e_i is the vector which is 1 in position i and 0 elsewhere. Show that $e_1, \ldots, e_n$ form a basis for $\mathbb{R}^n$ (5 points).*

A basis is a set which is both linearly independent and spans the vector space.

First, we show linear independence. Suppose that $c_1 e_1 + \cdots + c_n e_n = 0$. Since $c_i e_i$ is the vector which is c_i in the ith position and 0 elsewhere, $c_1 e_1 + \cdots = c_n e_n = (c_1, c_2, \ldots, c_n)$. Thus, $c_1 = c_2 = \cdots = c_n = 0$, so $e_1, \ldots, e_n$ are linearly independent.

Second, we show that the vectors span $\mathbb{R}^n$. If $v = (c_1, \ldots, c_n)$ is any vector in $\mathbb{R}^n$, then $v = c_1 e_1 + \cdots + c_n e_n$, as stated above. Thus, $e_1, \ldots, e_n$ span $\mathbb{R}^n$.