1 *Find a general solution (6 points) for the system*

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad A = \begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix}.$$

Find the solution with initial conditions $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ *(4 points).*

The characteristic polynomial of A is:

$$\begin{vmatrix} 1-\lambda & 3 \\ 12 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 36 = \lambda^2 - 2\lambda - 35 = (\lambda - 7)(\lambda + 5).$$

Thus, the eigenvalues are 7 and -5. The 7-eigenvectors are the null space of the matrix:

$$\begin{bmatrix} -6 & 3 \\ 12 & -6 \end{bmatrix},$$

and a basis for the eigenspace is $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$.

The -5-eigenvectors are the null space of the matrix:

$$\begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix}$$

and a basis for the eigenspace is $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$.

Putting this together, the general solution is

$$\mathbf{x}(t) = C_1 e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For any real numbers C_1 and C_2.

The value of $\mathbf{x}(0)$ is

$$\begin{bmatrix} C_1 + C_2 \\ 2C_1 - 2C_2 \end{bmatrix}.$$

This will be $\begin{bmatrix} 2 & 0 \end{bmatrix}^T$ when $C_1 = C_2 = 1$. Thus, the solution with the given initial conditions is

$$\mathbf{x}(t) = \begin{bmatrix} e^{7t} + e^{-5t} \\ 2e^{7t} - 2e^{-5t} \end{bmatrix}$$