1 *Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data points:* $(1,0)$, $(2,1)$, $(4,2)$, $(5,3)$. *(5 points)*

We have the matrices:

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

The normal equations are:

$$X^T X\beta = X^T y$$

$$\begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}$$

The determinant of the matrix on the left is $4 \cdot 46 - 12^2 = 184 - 144 = 40$. Thus, the solution is

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 46 & -12 \\ -12 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 25 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} -24 \\ 28 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 7/10 \end{bmatrix}.$$

So, the line of best-fit is $y = \frac{-3}{5} + \frac{7}{10}x$

2 *Orthogonally diagonalize the matrix*

$$A = \begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}.$$

Its eigenvalues are 13, 7, *and* 1 *(5 points).*

For the eigenvalue 13, we have

$$\begin{bmatrix} -6 & -4 & 4 \\ -4 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ -6 & -4 & 4 \\ 4 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & 4 \\ 0 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

A 13-eigenvector is $\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T$. For $\lambda = 7$,

$$\begin{bmatrix} 0 & -4 & 4 \\ -4 & -2 & 0 \\ 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -4 & 4 \\ 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A 7-eigenvector is $\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T$. For the eigenvalue 1,

$$\begin{bmatrix} 6 & -4 & 4 \\ -4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 2/3 \\ 0 & 4/3 & 8/3 \\ 0 & 8/3 & 16/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A 1-eigenvector is $\begin{bmatrix} -2 & -2 & 1 \end{bmatrix}$. Note that our three eigenvectors are orthogonal. Each of them has length 3, so we normalize them and make them the columns of P:

$$P = \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 13 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and then $A = PDP^{-1}$.