1 *Find an invertible matrix P and a matrix C such that*

$$\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} = PCP^{-1} \quad \textit{where } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(4 points)

The characteristic equation is

$$\begin{vmatrix} 5-\lambda & -5 \\ 1 & 1-\lambda \end{vmatrix} = 5 - 6\lambda + \lambda^2 + 5 = \lambda^2 - 6\lambda + 10.$$

By the quadratic formula, its roots are:

$$\frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i.$$

Choosing the "+" root, we have that

$$C = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}.$$

To find, P, we need to find a corresponding eigenvector, which is an element of the null space of the matrix

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}.$$

By looking at the second row, we can see that

$$\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

is an eigenvector. Taking the real and imaginary parts of this matrix, we have

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

2 *Find an orthonormal basis for the vector space spanned by the vectors: (6 points)*

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 3 \end{bmatrix}.$$

We set $u_1 = v_1$. Then we compute:

$$\begin{aligned} v_2 \cdot u_1 &= 0 - 2 + 0 + 0 = -2, \\ u_1 \cdot u_1 &= 1 + 1 + 0 + 0 = 2. \end{aligned}$$

Thus,

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

To find u_3, we compute

$$\begin{aligned} v_3 \cdot u_1 &= 4 - 2 + 0 + 0 = 2, \\ v_3 \cdot u_2 &= 4 + 2 + 0 + 3 = 9, \\ u_2 \cdot u_2 &= 1 + 1 + 0 + 1 = 3. \end{aligned}$$

Thus,

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

The vectors u_1, u_2, and u_3 form an orthogonal basis. To find an orthonormal basis, we divide each by the square root of its dot product with itself to get the basis:

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$