Math 54 quiz solutions October 14, 2009

1 *Find matrices P and D, with D diagonal, such that: (4 points)*

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = PDP^{-1}.$$

The characteristic equation of the matrix is

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 2 - 3\lambda + \lambda^2 - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2).$$

(The factorization was found by looking for factors of -10 which added to -3. I could have also used the quadratic formula.) Thus the eigenvalues are 5 and -2. For the eigenvalue 5, the eigenvectors are the null space of the matrix:

$$\begin{bmatrix} 2-5 & 3 \\ 4 & 1-5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix},$$

which is obviously the vector space spanned by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For, the eigenvalue 2, the eigenvectors are the null space of:

$$\begin{bmatrix} 2-(-2) & 3 \\ 4 & 1-(-2) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix},$$

which is the vector space spanned by

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Thus, we have

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

2 *Define* eigenvalue *(2 points).*

An eigenvalue of a matrix A is a scalar λ such that $Av = \lambda v$ for some non-zero vector v.

3 *Consider the bases*

$$\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}.$$

Find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and the matrix from $\mathcal{C}$ to $\mathcal{B}$. Be sure to indicate which is which (4 points).

The change of coordinates matrices from $\mathcal{B}$ and $\mathcal{C}$ to the standard basis are, respectively,

$$P_{\mathcal{B}} = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} \qquad\qquad P_{\mathcal{C}} = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}.$$

Their determinates are 8 and -8 respectively. We can find their inverses using the adjugate formula:

$$P_{\mathcal{B}}^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 \\ -5 & 7 \end{bmatrix} \qquad\qquad P_{\mathcal{C}}^{-1} = \frac{1}{-8}\begin{bmatrix} 2 & 2 \\ 5 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -2 & -2 \\ -5 & -1 \end{bmatrix}.$$

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ can be obtained by first changing from $\mathcal{B}$ to the standard basis and then to $\mathcal{C}$:

$$P_{\mathcal{C}}^{-1}P_{\mathcal{B}} = \frac{1}{8}\begin{bmatrix} -2 & -2 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -24 & 8 \\ -40 & 16 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}.$$

Similarly, the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$P_{\mathcal{B}}^{-1}P_{\mathcal{C}} = \frac{1}{8}\begin{bmatrix} -1 & 3 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16 & 8 \\ -40 & 24 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

Notice that the two change of coordinates matrices are inverses of each other.