1 *Find the determinant of*

$$A = \begin{bmatrix} h & 3 & 2 \\ 3 & -6 & 1 \\ -4 & 5 & -1 \end{bmatrix}.$$

(4 points). For what values of h is A invertible (2 points)?

We expand by minors along the first row:

$$\begin{aligned} \det A &= h\begin{vmatrix} -6 & 1 \\ 5 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ -4 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -6 \\ -4 & 5 \end{vmatrix} \\ &= h(6-5) - 3(-3+4) + 2(15-24) \\ &= h - 3 - 18 = h - 21 \end{aligned}$$

A is invertible if and only if h is not equal to 21.

It is also possible to find the determinant by row reduction. In this case you need to switch the first and third columns. This is not an elementary row operation, but it only changes the determinant by a factor of -1. You can't have h as a pivot entry because you don't know if it is equal to 0.

2 *Find the inverse of the following matrix (4 points):*

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

We put form the matrix $[A \mid I_3]$ and row reduce it:

$$\begin{bmatrix} 3 & -4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 3 & -4 & 2 & 1 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & -4 & 1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & -1 & -4 & 1 & 0 & -3 \\ 0 & 2 & 5 & 0 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 4 & -1 & 0 & 3 \\ 0 & 2 & 5 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & -1 & 0 & 4 \\ 0 & 1 & 4 & -1 & 0 & 3 \\ 0 & 0 & -3 & 2 & 1 & -6 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 6 & -1 & 0 & 4 \\ 0 & 1 & 4 & -1 & 0 & 3 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 2 & -8 \\ 0 & 1 & 0 & \frac{5}{3} & \frac{4}{3} & -5 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & 2 \end{bmatrix}$$

Thus, the inverse is:

$$A^{-1} = \begin{bmatrix} 3 & 2 & -8 \\ \frac{5}{3} & \frac{4}{3} & -5 \\ -\frac{2}{3} & -\frac{1}{3} & 2 \end{bmatrix}$$

This can also be found by using the adjugate formula.

3 ***Bonus Problem (1 point)*** *Suppose A is an $n \times n$ matrix with a 1-dimensional null space. What is the dimension of the columns space of* $\operatorname{adj}(A)$ *and why?*

Even when A is not invertible, the formula $A \cdot \operatorname{adj}(A) = \det A \cdot I_n$ still holds. If A has a 1-dimensional null space, then A is not invertible, so $\det A = 0$. Thus, the formula tells us $A \cdot \operatorname{adj}(A) = 0$, so every column of $\operatorname{adj}(A)$ is in the null space of A. But the null spaces of A is 1-dimensional, so the column space of $\operatorname{adj}(A)$ has dimension 1.