1. *Describe the solutions to the following in parametric vector form: (3 points)*

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 2 & 1 & 2 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We use elementary row operations to row reduce the matrix:

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 2 & 1 & 2 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & -2 & 1 & -9 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -7 \end{bmatrix}$$

The row operations are:

1. Swap the first and third rows.
2. Add -2 times the first row to the second and -3 times the first row to the third.
3. Multiply the second row by -1.
4. Add 2 times the second row to the third row.
5. Add -1 times the third row to the first.
6. Add -1 times the second row to the first.

From this we can see that x_4 is the free variable. Using t to denote the value of x_4, the parametric solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -9 \\ -1 \\ 7 \\ 1 \end{bmatrix}$$

2. *Describe the solutions to the following equation in parametric vector form: (3 points)*

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 2 & 1 & 2 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

In the previous problem, we found all solutions to the homogeneous system of equations. To solve the inhomogeneous system, we only need to find one solution and then the rest will be formed by adding a solution to the homogeneous equation. Since the right hand side is just the third column of the matrix, we can easily figure out one solution. The general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ -1 \\ 7 \\ 1 \end{bmatrix}$$

Alternatively, you could row reduce the augmented matrix and get the same solution.

3. *Are the vectors*

$$\begin{bmatrix} -4 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 4 \\ 3 \\ 6 \end{bmatrix}$$

linearly independent? (4 points)

Row reduce:

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

The operations are:

1. Swap the first and third rows.
2. Add 4 times the first row to the third and -5 times the first row to the fourth.
3. Add -3 times the second row to the third and 4 times the secon row to the fourth.

Since every column has a pivot entry, the columns are linearly independent.