1 *Find a general solution to the system: (5 points)*

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The characteristic equation of the matrix is:

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} = 2 - 3\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1),$$

so the eigenvalues are 4 and -1. For $\lambda = 4$,

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \qquad \text{eigenvector } \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

For $\lambda = -1$,

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \qquad \text{eigenvector } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, the solution to the homogeneous equation is

$$\mathbf{x}_h(t) = \begin{bmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

for any real numbers C_1 and C_2.

For the inhomogeneous equation, because the inhomogeneity is constant, we guess $\mathbf{x}_p(t) = \mathbf{a}$ for some vector $\mathbf{a}$. We plug this guess into the differential equation:

$$\mathbf{0} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{a} + \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \mathbf{a}.$$

To solve this linear system of equations, we form the augmented matrix and use Gaussian elimination:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Thus, $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The general solution is

$$\mathbf{x}(t) = \begin{bmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

2 *Find the Fourier series for the function on $-\pi < x < \pi$ defined by: (5 points)*

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Note that f is an odd function. Thus,

$$\frac{1}{\pi}\int_{-1}^{1} f(x)\cos(nx)\,dx = 0$$

for all n. Using the fact that the integral of an even function from -1 to 1 is twice the integral from 0 to 1, the coefficient of $\sin(nx)$ is

$$\begin{aligned} \frac{1}{\pi}\int_{-1}^{1} f(x)\sin(nx)\,dx &= \frac{2}{\pi}\int_{0}^{1} \sin(nx)\,dx \\ &= \left[\frac{-2}{n\pi}\cos(nx)\right]_0^1 \\ &= \frac{-2}{n\pi}((-1)^n - 1) = \frac{2(1-(-1)^n)}{n\pi}, \end{aligned}$$

which is $4/n$ when n is odd and 0 when n is even. We can the Fourier series as

$$\sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi}\sin(nx).$$

However, it is somewhat nicer to just sum over n odd by letting $n = 2m-1$ and then writing the Fourier series as,

$$\sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin\big((2m-1)x\big).$$