

1 Find all solutions to the system of equations:

$$\begin{aligned}x_1 - 4x_2 + 7x_3 &= 5 \\x_2 - 4x_3 &= 3 \\2x_1 - 6x_2 + 6x_3 &= -4\end{aligned}$$

We form the augmented matrix and use row operations to put it in reduced echelon form:

$$\begin{bmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & 5 \\ 0 & 1 & -4 & 3 \\ 0 & 2 & -8 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & -20 \end{bmatrix}$$

The steps are: first, we add -2 times the first row to the last row; and, second, we add -2 times the second row to the third row. Since there is a row which is zero except in the last column, the system is inconsistent. There are no solutions.

2 Find all solutions to the system of equations:

$$\begin{aligned}x_2 - 2x_3 &= 0 \\x_1 + 4x_2 - 5x_3 &= -2 \\3x_1 + 4x_2 + x_3 &= -3\end{aligned}$$

We use row operations to put the augmented matrix in echelon form:

$$\begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 4 & -5 & -2 \\ 3 & 4 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & -2 \\ 0 & 1 & -2 & 0 \\ 3 & 4 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & -8 & 16 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -5 & -2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

where the row operations are: first, swap the first two rows; second, add -3 times the first row to the third row; third, add 8 times the second row to the third row. The last row of the resulting matrix corresponds to the equation $0 = 3$, so the system of equations has no solution.