MATH 380A/500A, PROBLEM SET 6

These problems are due at the beginning of class on Monday, October 15.

(1) Eisenbud, exercise 5.3.
(2) Eisenbud, exercise 5.5.
(3) Let R be a local Noetherian ring and M a finitely generated R-module. Suppose n is the minimal number of generators of M. Show that if $\phi\colon R^n \to M$ and $\psi\colon R^n \to M$ are two surjections, then there exists an isomorphism $\alpha\colon R^n \to R^n$ such that $\phi = \psi \circ \alpha$. (Hint: Nakayama's Lemma.)

Now suppose we have two free resolutions of M:

$$\cdots \to F_2 \to F_1 \to F_0 \to M$$

and

$$\cdots \to F'_2 \to F'_1 \to F'_0 \to M$$

such that each F_i is chosen to have the minimal number of generators needed to surject onto the appropriate kernel. Such a resolution is called a *minimal free resolution*. Prove that there exist isomorphisms $\alpha_i\colon F_i \to F'_i$ which commute with the boundary maps:

$$\begin{array}{ccc} F_i & \longrightarrow & F_{i-1} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i-1}} \\ F'_i & \longrightarrow & F'_{i-1} \end{array}$$

(4) Let R be a local Noetherian ring, M a finitely generated R-module, and F a minimal free resolution as in the previous problem. Prove that if $F_i \cong R^{n_i}$, then $\operatorname{Tor}_i^R(M,k) = k^{n_i}$, where k is the quotient of R by its unique maximal ideal.
(5) Let $R = k[x]/\langle x^2\rangle$, where k is a field. We consider k as an R-module via the quotient $k = R/\langle x\rangle$. Prove that $\operatorname{Tor}_i^R(k,k) = k$ for all $i \geq 0$.
(6) Eisenbud, exercise 6.4. (You can use exercise 3.4 as in the hint without proving it.)
(7) Eisenbud, exercise 6.8.