MATH 380A/500A, PROBLEM SET 3

The following problems are due at the beginning of class on Sep. 23, 2013. In these problems, all rings are assumed to be Noetherian.

(1) Prove that if I is an ideal in a ring R and the radical of I is a maximal ideal, then I is primary.
(2) Exercise 3.5.
(3) Exercise 3.6.
(4) Compute a minimal primary decomposition of the ideal $\langle x^2y, xy^2\rangle$ in the ring $k[x, y]$, where k is a field. Which associated primes are embedded?
(5) Let k be any field and let $R = k[x, y, z]/\langle xy - z^2\rangle$. Prove that $P = \langle x, z\rangle$ is a prime ideal in R. (Hint: what is R/P?) Show that P^2 is, however, not primary.
(6) Exercise 1.8.
(7) Exercise 1.24.
(8) If a, b, c, and d are elements of a field k, when is the product of linear polynomials

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

equal to zero in $k[x]$? Use the answer to the previous question and the Nullstellensatz to compute the minimal primes of the ideal $I = \langle ac, ad + bc, bd\rangle \subset k[a, b, c, d]$. Is I radical?
(9) Find and correct any mistakes in the previous problems.