MATH 380A/500A, PROBLEM SET 2

The following problems are due at the beginning of class on Sep. 16, 2013:

(1) Suppose we have an exact sequence of R-modules:

$$0 \to M' \to M \to M'' \to 0$$

Prove that M is Noetherian if and only if both M' and M'' are Noetherian. Do the same for being Artinian and for having a composition series.

(2) Suppose that M is a Noetherian R-module and $\phi\colon M \to M$ is a surjective R-homomorphism. Prove that ϕ is an isomorphism. (Hint: consider the chain $\ker\phi = \phi^{-1}(0) \subset \phi^{-1}(\phi^{-1}(0)) \subset \cdots$.)

(3) Let M be a Noetherian R-module such that the annihilator of M is trivial. Prove that R is Noetherian.

(4) Suppose R is a ring which contains a field k, which means that R is a k-vector space. Further suppose that R has dimension 2 as a vector space. Prove that R is an Artinian ring and isomorphic to one of the following[1]:
- A degree 2 field extension k' of k.
- A direct product $k \times k$
- The quotient $k[x]/\langle x^2 \rangle$.

(5) Exercise 3.2.

(6) Exercise 3.17a.

(7) Exercise 3.20.

(8) Exercise 3.7.

[1]The first of these possibilities was added in a correction on Sept. 13