MATH 380A/500A, PROBLEM SET 1

The following problems are due at the beginning of class on Sep. 9, 2013:

(1) Eisenbud, exercise 2.1.
(2) Eisenbud, exercise 2.3.
(3) Eisenbud, exercise 2.4.
(4) Eisenbud, exercise 2.6.
(5) The usual Chinese remainder theorem states if $m_1, \ldots, m_n$ are pairwise coprime integers and $a_1, \ldots, a_n$ are arbitrary, then there exists an integer b such that $b \equiv a_i \pmod{m_i}$ for all i. Prove this version of the Chinese remainder theorem using the previous problem.
(6) Eisenbud, exercise 2.10, including the last sentence.
(7) Eisenbud, exercise 2.26. Nontrivial idempotents are defined in exercise 2.25.
(8) Let f be an element of a ring R. Let $U = \{1, f, f^2, \cdots\}$ be the set of all powers of f. Prove that $R[U^{-1}]$ is isomorphic to $R[x]/\langle fx - 1\rangle$. We will write $R[f^{-1}]$ for this localization.
(9) Let I be a partially ordered set and assume that for each i and j in I, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. Suppose that for each $i \in I$, we have a ring R_i and for each $i < j$, we have a ring homomorphism $\phi_{ij} \colon R_i \to R_j$ such that for $i < j < k$, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. For $i = j$, we adopt the convention that $\phi_{ii} \colon R_i \to R_i$ is the identity. We call this set-up a *filtered system of rings.*

    We define the *colimit* $\varinjlim R_i$ (which really depends not just on the rings R_i but also on the homomorphisms ϕ_{ij}) to be disjoint union of the R_i modulo the equivalence relation that $r_i \in R_i$ and $r_j \in R_j$ are equivalent if there exists $k \geq i, j$ such that $\phi_{ik}(r_i) = \phi_{jk}(r_j)$. Describe the natural addition and multiplication operations on these equivalence classes and verify that it makes the colimit into a ring.

    (The assumption on the indexing set makes this a *filtered colimit.* Colimits can also be constructed without the filtered hypothesis. See Appendix 6 for details.)
(10) With the same set-up as in the previous problem, further suppose that that the index set I has a greatest element m (i.e. $i \leq m$ for all $i \in I$). Show that $\varinjlim R_i \cong R_m$.
(11) Let U be a multiplicative set. We let I consist of the elements of U modulo multiplication by units, and we order it by divisibility, so that if f and g are elements of U, then $f \leq g$ if there exists $h \in R$ such that $g = fh$. We set $R_f = R[f^{-1}]$ as in problem 8 and $\phi_{fg} \colon R[f^{-1}] \to R[g^{-1}]$ by $\phi_{fg}(r/f^n) = rh^n/g^n$. Show that we have constructed a well-defined filtered system of rings. Show that the colimit $\varinjlim R_f$ is isomorphic to $R[U^{-1}]$.