MATH 380A/500A, FINAL EXAM

Rings are not necessarily assumed to be Noetherian unless stated.

Problem 1

A ring is called semilocal *if it has finitely many maximal ideals. Let R be a semilocal ring and let M and N be finitely presented R-modules. Show that if $M_P \cong N_P$ for all maximal ideals P, then $M \cong N$ (20 points). (See Eisenbud, Exercise 4.13 and its hint.)*

We let $P_1, \ldots, P_n$ be the maximal ideals of R. By assumption, we have an isomorphism $M_{P_i} \cong N_{P_i}$, and then by Proposition 2.10, we can identify our isomorphism with an element of $\mathrm{Hom}(M, N) \otimes R_P$. By clearing denominators, we get a morphism $\phi_i \colon M \to N$ such that $(\phi_i)_{P_i} \colon M \to N$ is an isomorphism.

Since they are each maximal, the P_i are pairwise disjoint and prime, and so by Prime Avoidance (Lemma 3.3), we may choose an element $a_i \in P_i$ such that $a_i \notin P_j$ for all $j \neq i$. Now set $b_i = \prod_{j \neq i} a_j$ and then we have that $b_i \notin P_i$, but $b_i \in P_j$ for all $j \neq i$. We set ψ to be the homomorphism $\sum_j b_j \phi_j$, and we will show that ψ is an isomorphism.

By Corollary 2.9, it suffices to show that for each maximal P_i, the localization ψ_{P_i} is an isomorphism and by Corollary 4.4, it suffices to show that ψ_{P_i} is a surjection. We consider the composition of ψ_{P_i} with the quotient map $\pi_i \colon N_{P_i} \to N_{P_i}/P_i N_{P_i}$. Since $a_j \in P_i$ for $j \neq i$, $\pi_i \circ \psi_{P_i} = \pi_i \circ (\phi_i)_{P_i}$ is surjective. Therefore, by Nakayama's Lemma, ψ_{P_i} is surjective.

Finally, since $(\phi_i)_{P_i}$ is an isomorphism $(\phi_i)_{P_i}^{-1} \circ \psi_{P_i}$ is also surjective, but then by Corollary 4.4, it must be an isomorphism. Thus, ψ_{P_i} is also an isomorphism for each P_i and so ψ is an isomorphism.

Problem 2

Let R be a complete discrete valuation ring with fraction field K and let L be an algebraic extension of K. If S is the integral closure of R in L, prove that S is a valuation ring (20 points). (Hint: you may find Eisenbud, Corollary 7.6 of use.) Second, if, additionally, L is the algebraic closure of K, prove that the valuation group of S is isomorphic to $\mathbb{Q}$ (10 points).

Let x be an element of L and we will prove that either x or x^{-1} is in S. Since L is an algebraic extension of K, the field L' generated by x is finite. Let S' be the integral closure of R in L', and we now verify that S' is a discrete valuation ring. By the Krull-Akizuki theorem, S' is Noetherian. By Eisenbud, Proposition 9.2, S' is 1-dimensional. In order to show that S' is a DVR, it only remains to prove that it is local, since S' is integrally closed by construction.

Let $\mathfrak{m} \subset S'$ be a maximal ideal contracting to the maximal ideal of R, which exists by the Lying Over theorem. Let $s_1, \ldots, s_n$ be the generators of $\mathfrak{m}$ and let r be an element of S' not in $\mathfrak{m}$. Then consider the subring of S' generated by the elements $s_1, \ldots, s_n, r$ and by R. We denote this by S'', and it is finite as an R-module. Therefore, by Eisenbud, Corollary 7.6, S'' is a product of local domains. But $S'' \subset S \subset L'$, and L' is a field, so S'' is a local domain. By the Going Up theorem, $\mathfrak{m} \cap S''$ is a maximal ideal. Thus, r is a unit. Therefore, $\mathfrak{m}$ must be the unique maximal ideal of S', so it is local.

Therefore, S' is a DVR, and so either x or x^{-1} is in S', and thus one of them is in S, which is what we wanted to show.

Now, suppose that L is the algebraic closure of K, and we claim that the valuation group of S is isomorphic to $\mathbb{Q}$. First of all, we recall that the valuation group of a valuation ring R can be written as the quotient of multiplicative groups K^*/R^*, where K is the fraction field of R. From this it is immediate, that any inclusion of valuation rings $R \subset S$ induces a homomorphism of valuation groups. Furthermore, if $R \subset S$ is an integral extension, then a non-unit $r \in R$ is contained in some maximal ideal of S by the Lying Over theorem, and thus r is not a unit in S. Therefore, for integral extensions of valuation rings, the homomorphism of valuation groups is injective.

If we let S' be the integral closure in a finite algebraic extension as above, then valuation group of S' is isomorphic to $\mathbb{Z}$, since S' is a DVR. By the previous paragraph, the valuation group of R is the subgroup $d\mathbb{Z}$ for some positive integer d. We rescale so that the inclusion is identified with $\mathbb{Z} \subset \frac{1}{d}\mathbb{Z}$. Thus, we can identify the valuation of any element of S with an element of $\mathbb{Q}$, which is our desired isomorphism. To show that this identification is surjective, we let π be a uniformizer of R and then consider L defined by adjoining $\pi^{1/d}$, which will have valuation $1/d$.

Problem 3

Let R be a Dedekind domain and M an R-module. Prove that M is flat if and only if M has no associated primes other than the zero ideal (20 points).

Suppose that $\mathfrak{m}$ is an associated prime of M, say because $\mathfrak{m} = \operatorname{ann}(m)$ for $m \in M$. Then any $r \in \mathfrak{m}$ is a zerodivisor and M is not flat by Corollary 6.3.

Conversely, suppose that (0) is the only associated prime of M and we want to show that M is flat. Let $N' \to N$ be the inclusion of R-modules and consider the tensor product $M \otimes N' \to M \otimes N$. By Corollary 2.9, it suffices to check injectivity locally at the maximal ideals of $\mathfrak{m}$. However, for any maximal ideal $\mathfrak{m}$, the localization $R_{\mathfrak{m}}$ is a DVR and thus a principal ideal domain, and since associated primes localize, $M_{\mathfrak{m}}$ has no non-zero associated primes. Thus, $M_{\mathfrak{m}}$ is torsion-free, so $M_{\mathfrak{m}}$ is flat by Corollary 6.3. Thus, $M_{\mathfrak{m}} \otimes N'_{\mathfrak{m}} \to M_{\mathfrak{m}} \otimes N_{\mathfrak{m}}$ is injective and M is flat.

Problem 4

Let k be a field and $R = k[x,y]/\langle x^2 - x, xy\rangle$. Find a k-subalgebra $S \subset R$ such that $S \cong k[z]$ and R is a finite S-module, as guaranteed by Noether normalization (10 points). Second, prove that there can exist no such S where R is flat as an S-module (20 points).

We take $S = k[y]$ and then R is a finite S-module. In order to show that S is a subring of R, i.e. that there are no relations imposed on y, we show that $x^2 - x, xy$ is a Gröbner basis for the lexicographic order with $x > y$. There is a single S-polynomial to compute:

$$y(x^2 - x) - x(xy) = -xy,$$

which immediately reduces to zero using the second polynomial. Therefore, the kernel of $k[y] \to R$ is zero, so we have a subring.

To see that R is a finite S-module, we can use the relation $x^2 - x$ together with the elements 1 and x to generate R.

Because R has dimension 1, any S such that R is a finite S-algebra must be 1-dimensional. Now suppose we have a subalgebra $S \cong k[z]$ of R. If we consider $x \in R$, then the annihilator of x is $\mathfrak{m} = \langle x - 1, y\rangle$. Thus, $\mathfrak{m}$ is an associated prime of R and since $R/\mathfrak{m}$ is isomorphic to a field k, $\mathfrak{m}$ has dimension 0. Therefore, by Proposition 9.2, $\mathfrak{m} \cap S$ is also zero-dimensional, so $\mathfrak{m} \cap S$ is a non-zero ideal. Therefore, $\operatorname{ann}_S(f)$ is non-zero, so R is not a flat S-module by Corollary 6.3.

(If R is any affine ring, then there exists a flat Noether normalization if and only if all Noether normalizations are flat. The existence of a flat Noether normalization is equivalent to being both equidimensional and Cohen-Macaulay. The necessity of R being equidimensional was illustrated by this exercise. Cohen-Macaulay is a more subtle condition which is discussed in the later chapters of Eisenbud.)

Problem 5

Let k be a field and $R = k[x_1, \ldots, x_r]$, which, as usual, is graded by degree. If I is a homogeneous ideal in R and $>$ is any term order, show that I has a homogeneous Gröbner basis (20 points). Give an example of a homogeneous ideal which has a non-homogeneous Gröbner basis (10 points).

First proof: Let x^a be a generator of $\text{in}_>(I)$, so that $x^a = \text{in}_>(f)$ for some $f \in I$. We can write $f = f_1 + \cdots + f_n$, where each of the f_i are homogeneous, and since I is homogeneous, the f_i are also in i. Then we take whichever f_i contains x^a as one of the elements of our Gröbner basis, and this is homogeneous and clearly $\text{in}_>(f_i) = x^a$. If we do this for all generators of $\text{in}_>(I)$, we've constructed a homogeneous Gröbner basis.

Second proof: We start with a homogeneous generating set for I and we run Buchberger's algorithm to construct a Gröbner basis. At each step where we compute an S-polynomial $x^a f - x^b g$, the exponents a and b are chosen so that the leading terms of $x^a f$ and $x^b g$ are the same, so in particular they have the same degree. Thus, the S-polynomial is also homogeneous. Similarly, each reduction step preserves the homogeneity, so we will produce a homogeneous Gröbner basis.

For the last sentence, we can take the ideal $I = \langle x_1, x_2 \rangle$. Since I is already a monomial ideal, $\text{in}_>(I) = I$. However, we can also choose a non-homogeneous Gröbner basis such as $x_1 - x_2^2, x_2$, with a term order such that $x_1 > x_2^2$, such as the lexicographic order.