Math 1B Problems, volume 3

Dustin Cartwright[1]

1. Let $P(x) = 2 - 3x^2 + 5x^3$. What is $P^{(n)}(0)$ for $n = 1, 2, 3, 4, 5$? What is the Taylor series for P?

2. Show that the function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

3. Find the first three nonzero terms in the Maclaurin series of $f(x) = \sec x$.

4. Determine the Maclaurin series for each of the following by manipulating previously known series:

   (a) $\ln\left(\dfrac{1+x}{1-x}\right)$.

   (b) $\sin^2(x)$

   (c) e^{-x^2}

5. Use Maclaurin series to find $\displaystyle\lim_{x\to 0} \frac{(\sin x - x)^3}{x(1-\cos x)^4}$.

6. If you have a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with radius of convergence R, and a number a whose absolute value is less than R, what is the Taylor series of f centered at a?

7. Find an approximation to $E = \dfrac{q}{(D-d)^2} - \dfrac{q}{(D+d)^2}$ when d is much smaller than D by expanding in terms of powers of d/D. You should use at least 2 terms in the power series expansion.

8. Write each of the following statements about a function as a differential equation and give an example of a non-zero function which satisifies the statement:

   (a) Equals its derivative.

   (b) Equals the negative of its derivative.

   (c) Equals the negative of its second derivative.

[1]Problems borrowed from various sources, mostly the Math 1b workbook

9. Say you have \$100 invested in a bank account which pays 3% interest per year. The interest is compounded continuously, meaning that the rate that interest is being earned at every instant is proportional to the amount of money in the bank account at that instant. What is a differential equation for the money M in the bank account? What is an equation for M?

10. Check that $y = \tan x$ is a solution to $y' = 1 + y^2$.

11. Find a solution of $y' = \sqrt{1 - y^2}$. (Hint: think about where you've seen expressions like $\sqrt{1 - y^2}$ before in calculus or trigonometry.) What is the general solution to this differential equation?

12. Plot some solutions to the differential equation in the previous problem. Do the curves cross each other? Should they cross each other?

13. Suppose a baseball is dropped from a certain height. There are two forces acting on the baseball: gravity and air resistance. We're going to assume that these forces can be modelled with the differential equation:

$$\frac{dv}{dt} = g - bv^2$$

where g is the acceleration due to gravity and b is a coefficient of friction.

(a) Draw the direction field for this differential equation, assuming $g = b = 1$. Draw some solutions. What happens to these solutions as $t \to \infty$.

(b) Find the general solution to the differential equation. (Hint: partial fractions)

(c) Find the limit of velocity as $t \to \infty$ for this family of solutions. Does the limit depend on the value of the constant?

(d) Find the particular solution which satisifies the initial condition of $v = 0$ when $t = 0$.

(e) Using this, find an equation for the distance x that the baseball falls after time t.

(f) Find the particular with the initial condition $v = 2\sqrt{g/b}$ at $t = 0$.

14. Determine whether each of the following are separable, linear, both, or neither:

(a) $yy' = x\sqrt{1 + x^2}\sqrt{1 + y^2}$

(b) $y' + 2xy = 2x^3$

(c) $1 + 2xy^2 + 2x^2yy' = 0$

(d) $1 + y^2 - y'\sqrt{1 - x^2} = 0$

(e) $xy' - 2y = x^3$

(f) $y' = 2 + 2x^2 + y + x^2 y$

15. Solve the differential equation $y' = x - y$ and discuss the behavior of the solutions as $x \to \infty$ for various initial conditions. Draw a direction field and a few solutions if necessary.

16. Solve the differential equation $y' = -xy$ and discuss the behavior of the solutions as $x \to \infty$ for various initial conditions. Draw a direction field and a few solutions if necessary.

17. Use partial fractions to find a power series for $f(x) = \dfrac{3}{x^2 + x - 2}$

18. What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdots (2n)}$?

19. Suppose the series $\sum c_n x^n$ has radius of convergence 2 and $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n) x^n$?

20. Suppose that the radius of convergence of $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?