1 *If* $\sum_{n=0}^{\infty} c_n(x-3)^n$ *is a power series which converges absolutely for* $x=0$*, what can you say about its radius of convergence? (3 points) What if it converges conditionally for* $x=0$*? (2 points)*

This is a power series centered at $x=3$. The point $x=0$ is a distance of 3 from the center.

In the first case, you know that the radius of convergence R is at least 3, i.e. $R \geq 3$, because $x=0$ is in the interval of convergence.

In the second case, you know that the radius of convergence is exactly 3, because if it were less than 3, the series would diverge at $x=0$, and if it were more than 3, the series would converge absolutely at $x=0$. Thus, the only possibility left is that the radius is exactly 3.

2 *Find the Maclaurin series for* $f(x) = 2\sin x$ *using the definition of the Maclaurin series. (5 points)*

The derivatives of $f(x)$ are:

$$\begin{aligned} f^{(4n)}(x) &= 2\sin x \\ f^{(4n+1)}(x) &= 2\cos x \\ f^{(4n+2)}(x) &= -2\sin x \\ f^{(4n+3)}(x) &= -2\cos x \end{aligned}$$

for any integer n. Another way of saying this is that the derivatives just cycle through the four functions above in the order listed. We can evaluate each of these functions at $x=0$ to get:

$$\begin{aligned} f^{(4n)}(x) &= 0 \\ f^{(4n+1)}(x) &= 1 \\ f^{(4n+2)}(x) &= 0 \\ f^{(4n+3)}(x) &= -1 \end{aligned}$$

Thus, the Maclaurin has only terms of odd degree and these have alternating sign:

$$\frac{2x}{1!} - \frac{2x^3}{3!} + \frac{2x^5}{5!} - \frac{2x^7}{7!} + \cdots$$

We can rewrite this in sigma notation as:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+1}}{(2n+1)!}$$

3 *Express* $\dfrac{x}{x-2}$ *as a power series. (5 points)*

First rearrange the equation to get something looking more like the geometric series and then use the power series expansion of the geometric series:

$$\begin{aligned}
\frac{x}{x-2} &= \frac{-x}{2}\frac{1}{1-x/2} \\
&= \frac{-x}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{-x}{2}\frac{x^n}{2^n} \\
&= \sum_{n=0}^{\infty}\frac{-1}{2^{n+1}}x^{n+1} \\
&= \sum_{n=1}^{\infty}\frac{-1}{2^n}x^n
\end{aligned}$$

where the last step is reindexing so that the power of x is n instead of $n+1$.