Math 1B quiz solutions October 20, 2006

1 Does $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n \cdot 3^{n+1}}$ converge or diverge? (5 points)

All the terms are positive, so we can use a limit comparison test with $b_n = 1/n$:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{10^n/(n \cdot 3^{n+1})}{1/n} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{10}{3}\right)^n$$

which diverges to ∞ because $10/3$ is greater than 1. The series $\sum b_n$ diverges because it is the harmonic series, so $\sum a_n$ diverges as well.

2 Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}(x+1)^n$. (5 points) Find its interval of convergence. (5 points)

Use the ratio test to find the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}/(n+1)^2(x+1)^{n+1}}{(-1)^n/n^2(x+1)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{-n^2(x+1)}{n^2+2n+1} \right| \\
&= \lim_{n\to\infty} \frac{|x+1|}{1+2/n+1/n^2} \\
&= |x+1|
\end{aligned}$$

So, the series converges whenever $|x+1| < 1$ and diverges whenever $|x+1| > 1$. Thus, the radius of convergence is 1.

The center of the power series is $x = -1$. To find the interval of convergence, we need to check for convergence on the boundaries of the radius of convergence $x = -1 - 1 = -2$ and $x = -1 + 1 = 0$. If we plug in $x = 0$, the power series becomes:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

which is an alternating series and converges by the alternating series test. If we plug in $x = -2$, the power series becomes:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

which is a p-series with $p = 2$, so it converges. Thus, the interval of convergence is $[-2, 0]$.