Math 1B quiz solutions December 6, 2006

1 *Find a general solution to $y'' + 2xy' + 2y = 0$ as a power series (10 points).*

We write $y = \sum_{n=0}^{\infty} a_n x^n$ and then take the derivatives:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Then we substitute these into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$

In order to get the powers of x to all be the same, we replace n with $n+2$ in the first summation. Also, the term of the second summation with $n = 0$ is 0 so we can change the summation to start from 0 without changing the sum, so:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n = 0$$
$$\sum_{n=0}^{\infty} \big((n+2)(n+1) a_{n+2} + (2n+2) a_n\big) x^n = 0$$

If a power series is identically zero, then all the coefficients are zero, so

$$(n+2)(n+1) a_{n+2} + (2n+2) a_n = 0$$

for $n \geq 0$. Solving this for a_{n+2} gives the recursive formula:

$$a_{n+2} = -\frac{2n+2}{(n+2)(n+1)} a_n = -\frac{2}{n+2} a_n$$

The explicit formulas for a_n are:

$$a_{2k} = \frac{(-2)^k a_0}{(2k)(2k-2)\cdots(2)} = \frac{(-1)^k a_0}{k!}$$
$$a_{2k+1} = \frac{(-2)^k a_1}{(2k+1)(2k-1)\cdots(1)} = \frac{(-2)^k (2k)(2k-2)\cdots(2) a_1}{(2k+1)!} = \frac{(-1)^k 2^{2k} k! a_1}{(2k+1)!}$$

2 *A swinging pedulum can be approximately described by the differential equation:*

$$\theta'' + c\theta' + \frac{g}{L}\theta = 0$$

Suppose that $c = 2$, $g = 9.8$ and $L = 4.9$. What is the general solution to the differential equation? (5 points)

Substituting in the values of the constants, the equation becomes $\theta'' + 2\theta' + 2\theta = 0$. The auxillary equation is $r^2 + 2r + 2 = 0$. Using the quadratic formula the roots of the polynomial are:

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm 1i$$

Thus the general solution is:

$$\theta = Ae^{-t} \sin t + Be^{-t} \cos t$$