1 *Find the general solution to $y'' + 2y' + y = 5\sin 2x$ using undetermined coefficients (7 points).*

The auxillary equation is $r^2 + 2r + 1 = 0$ which has a root only at $r = -1$. Thus the complementary solution is $y = e^{-x} + xe^{-x}$.

We will guess $y_p = A\sin 2x + B\cos 2x$. Then the derivatives are:

$$y_p' = 2A\cos 2x - 2B\sin 2x$$
$$y_p'' = -4A\sin 2x - 4B\cos 2x$$

Then we plug these values into the differential equation to get:

$$\begin{aligned} y_p'' + 2y_p' + y_p =& (-4A\sin 2x - 4B\cos 2x) + (4A\cos 2x - 4B\sin 2x) \\ &+ (A\sin 2x + B\cos 2x) \\ =& (-3A - 4B)\sin 2x + (4A - 3B)\cos 2x \end{aligned}$$

Since this is supposed to equal $5\sin 2x$, then we have the following two equations:

$$-3A - 4B = 5$$
$$4A - 3B = 0$$

From the second we can solve and get $A = \frac{3}{4}B$. Then substitute this into the first equation to get:

$$\begin{aligned} -\frac{9}{4}B - 4B &= 5 \\ -\frac{25}{4}B &= 5 \\ B &= -\frac{4}{25}5 = -\frac{4}{5} \end{aligned}$$

and then $A = -\frac{3}{4}\frac{4}{5} = -\frac{3}{5}$.

Thus, the general solution is:

$$y = -\frac{3}{5}\sin 2x - \frac{4}{5}\cos 2x + C_1 e^{-x} + C_2 x e^{-x}$$

2 *Use variation of parameters to find the general solution to $y'' + y' - 2y = e^{2x}\cos x$ (8 points).*

(There was a mistake in this problem. You had the option of doing the problem as stated or changing the equation to either $y'' - y' - 2y = e^{2x}\cos x$ or to $y'' + y' - 2y = e^{-2x}\cos x$. It is certainly possible to solve the original equation, but it is more work than I intended. The solution will use the first of these alternate equations.)

The auxillary equation is $r^2 - r - 2 = 0$. This factors as $(r-2)(r+1)$, so it has roots $r = -1$ and $r = 2$. Thus the complementary solution is $y_c = C_1 e^{2x} + C_2 e^{-x}$.

To use variation of parameters, we try the solution $y_p = u_1 e^{2x} + u_2 e^{-x}$. Then we have the pair of equations:

$$\begin{aligned} u_1' e^{2x} + u_2' e^{-x} &= 0 \\ 2u_1' e^{2x} - u_2' e^{-x} &= e^{2x} \cos x \end{aligned}$$

From the first we can solve for u_1' to get $u_1' = -e^{-3x} u_2'$. Then substituting this into the second equation:

$$\begin{aligned} 2e^{-3x} e^{2x} u_2' - u_2' e^{-x} &= e^{2x} \cos x \\ -3e^{-x} u_2' &= e^{2x} \cos x \\ u_2' &= -\frac{1}{3} e^{3x} \cos x \end{aligned}$$

Then, from our previous equation for u_1', we have

$$u_1' = -e^{-3x} \left(-\frac{1}{3} e^{3x} \cos x \right) = \frac{1}{3} \cos x$$

We can solve this by taking the integral to find that $u_1 = \frac{1}{3} \sin x$.

To find u_2 we need to compute the integral $\int -\frac{1}{3} e^{3x} \cos x \, dx$. We use integration by parts with $u = -\frac{1}{3} e^{3x}$, $dv = \cos x \, dx$, and $du = -e^{3x} \, dx$, $v = \sin x$, so:

$$u_2 = \int -\frac{1}{3} e^{3x} \cos x \, dx = -\frac{1}{3} e^{3x} \sin x + \int e^{3x} \sin x \, dx$$

Using integration by parts again:

$$= -\frac{1}{3} e^{3x} \sin x + e^{3x} \cos x + \int 3e^{3x} \cos x \, dx$$

Now we have a multiple of our original integral, so we just move it over to the left side:

$$\begin{aligned} \int -\frac{10}{3} e^{3x} \cos x \, dx &= -\frac{1}{3} e^{3x} \sin x + e^{3x} \cos x \\ \int -\frac{1}{3} e^{3x} \cos x \, dx &= -\frac{1}{30} e^{3x} \sin x + \frac{1}{10} e^{3x} \cos x \end{aligned}$$

Putting everything together the particular solution is:

$$\begin{aligned} y_p &= -\frac{1}{3}e^{2x}\sin x + (-\frac{1}{30}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x)e^{-x} \\ &= (-\frac{1}{3} - \frac{1}{30})e^{2x}\sin x - \frac{1}{10}e^{2x}\cos x \\ &= -\frac{11}{30}e^{2x}\sin x - \frac{1}{10}e^{2x}\cos x \end{aligned}$$

The general solution is:

$$y = -\frac{11}{30}e^{2x}\sin x - \frac{1}{10}e^{2x}\cos x + C_1e^{2x} + C_2e^{-x}$$