Math 1B quiz solutions November 29, 2006

(Note: $\int \tan u\, du = \ln|\sec u| + C$)

1 *Find the general solution to $y'' + 4y' + 3y = e^{-x} + \sin x$ using the method of undetermined coefficients (7 points).*

The auxillary equation is $r^2 + 4r + 3 = 0$. This has solutions $r = -1$ and $r = -3$. Thus, the complementary solution is $y_c = C_1 e^{-x} + C_2 e^{-3x}$.

Our first guess for a solution might be $y_p = Ae^{-x} + B\sin x + C\cos x$, but notice that $Ae^{-x}$ is already a solution to the complementary equation. Thus, we multiply by $x$ to get $y_p = Axe^{-x} + B\sin x + C\cos x$. Now, we take the derivatives:

$$\begin{aligned} y_p' &= -Axe^{-x} + Ae^{-x} + B\cos x - C\sin x \\ y_p'' &= Axe^{-x} - 2Ae^{-x} - B\sin x - C\cos x \end{aligned}$$

Then we plug these into the original equation and group the similar terms:

$$\begin{aligned} y_p'' + 4y_p' + 3y_p =& (Axe^{-x} - 2Ae^{-x} - B\sin x - C\cos x) \\ &+ (-4Axe^{-x} + 4Ae^{-x} + 4B\cos x - 4C\sin x) \\ &+ (3Axe^{-x} + 3B\sin x + 3C\cos x) \\ =& (A - 4A + 3A)xe^{-x} + (-2A + 4A)e^{-x} \\ &+ (-B - 4C + 3B)\sin x + (-C + 4B + 3C)\cos x \\ =& 2Ae^{-x} + (2B - 4C)\sin x + (4B + 2C)\cos x \end{aligned}$$

Since this has to equal $e^{-x} + \sin x$, we can equate coefficients and get the following three equations:

$$\begin{aligned} 2A &= 1 \\ 2B - 4C &= 1 \\ 4B + 2C &= 0 \end{aligned}$$

This means that $A = 1/2$. Also $C = -2B$ from the last equation. Substituting this into the second equation gives $2B + 8B = 10B = 1$, so $B = 1/10$ and $C = -1/5$.

Putting all this together, the general solution is

$$y = \frac{xe^{-x}}{2} + \frac{\sin x}{10} - \frac{\cos x}{5} + C_1 e^{-x} + C_2 e^{-3x}.$$

2 *Find the general solution to $y'' + 4y = \sec 2x$. (Hint: use variation of parameters) (8 points).*

The auxillary equation is $r^2 + 4 = 0$, which has solutions $r = \pm 2i$. Thus the complementary solution is $y_c = C_1 \sin 2x + C_2 \cos 2x$.

Using variation of parameters, we guess $y_p = u_1 \sin 2x + u_2 \cos 2x$. Then we have the following two equations:

$$\begin{aligned} u_1' \sin 2x + u_2' \cos 2x &= 0 \\ 2u_1' \cos 2x - 2u_2' \sin 2x &= \sec 2x \end{aligned}$$

From the first, we get that

$$u_2' = -u_1' \sin 2x / \cos 2x = -u_1' \tan 2x$$

The substitute this into the second equation to get:

$$\begin{aligned} 2u_1' \cos 2x + 2u_1' \tan 2x \sin 2x &= \sec 2x \\ 2u_1'(\cos 2x + \sin 2x \tan 2x) &= \frac{1}{\cos 2x} \\ u_1' &= \frac{1}{2\cos 2x(\cos 2x + \sin 2x \tan 2x)} \\ &= \frac{1}{2(\cos^2 2x + \sin^2 2x)} \\ &= \frac{1}{2} \end{aligned}$$

Then we substitute to find $u_2'$:

$$u_2' = -u_1' \tan 2x = -\frac{1}{2} \tan 2x$$

Now we integrate $u_1'$ and $u_2'$ to get:

$$\begin{aligned} u_1 &= \frac{x}{2} \\ u_2 &= -\frac{1}{4} \ln|\sec 2x| \end{aligned}$$

Thus,

$$y_p = \frac{x \sin 2x}{2} - \frac{\ln|\sec 2x| \cos 2x}{4}$$

Putting this together with the complementary solution:

$$y = \frac{x \sin 2x}{2} - \frac{\ln|\sec 2x| \cos 2x}{4} + C_1 \sin 2x + C_2 \cos 2x$$