1 *Find the general solution to the equation* $y' + y/x + x^2 - 1 = 0$. *(5 points)*

This is a linear differential equation after rearranging to get the standard linear form:

$$y' + y/x = -x^2 + 1$$

First we find the antiderivative of $1/x$, which is $\ln x$. The integrating factor is then $e^{\ln x} = x$. Then we multiply by this integrating factor:

$$\begin{aligned} y'x + y &= (-x^2 + 1)x \\ (yx)' &= -x^3 + x \\ yx &= \int -x^3 + x \, dx \\ &= -\frac{x^4}{4} + \frac{x^2}{2} + C \\ y &= -\frac{x^3}{4} + \frac{x}{2} + \frac{C}{x} \end{aligned}$$

which is the general solution.

2 *Find the general solution to the differential equation* $4y'' + 4y' + y = 0$. *(5 points)*

This is a second-order differential equation. The auxillary polynomial is $4r^2 + 4r + 1$, which factors as $(2r+1)(2r+1)$. This has a root only at $r = -1/2$. Thus, the general solution is

$$C_1 e^{-x/2} + C_2 x e^{-x/2}$$

3 *Find the Maclaurin series for the function: (5 points)*

$$f(x) = \begin{cases} \frac{e^x - 1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

The Maclaurin series for $e^x - 1$ is

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Then we can divide by x term by term to get:

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$