Solutions to some limit convergence definition problems

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These are some possible questions which I came up with involving the definition of a convergent series, and solutions to these questions. The solutions are probably more detailed than you would need to write, on, for example, an exam, because I wanted to make sure that every step is as clear as possible.

A question about the definition of a convergent limit may or may not look like these, but perhaps reading these explanations helps you understand the idea of the definition.

1 *Use the definition of a convergent series to show that*

$$\sum_{i=1}^{\infty} \frac{1}{3^i} = \frac{1}{2}$$

First we're going to look at the partial sums:

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{3^i} \\ &= \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} \end{aligned}$$

There's a standard rule from algebra that a finite geometric series has the equation:

$$a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$$

We can use this formula with $r = 1/3$ and $a = 1/3$ to rewrite s_n:

$$\begin{aligned} s_n &= \frac{1}{3}\left(\frac{1-1/3^n}{1-1/3}\right) \\ &= \frac{1-1/3^n}{3-1} \\ &= \frac{1-1/3^n}{2} \\ &= \frac{1}{2}\left(1 - \frac{1}{3^n}\right) \end{aligned}$$

Now we have s_n in a much simpler form. In particular, we can see right away that as n gets bigger and bigger, $1 - 1/3^n$ will get closer and closer to 1 and so s_n will approach $1/2$. However, the problem asked us to show it using ε and N in the definition of limit.

The distance between the infinite sum and the partial sums is:

$$\begin{aligned} \left|s_n - \frac{1}{2}\right| &= \left|\frac{1}{2}\left(1 - \frac{1}{3^n}\right) - \frac{1}{2}\right| \\ &= \left|\frac{1}{2} - \frac{1}{2}\frac{1}{3^n} - \frac{1}{2}\right| \\ &= \left|-\frac{1}{2 \cdot 3^n}\right| \\ &= \frac{1}{2 \cdot 3^n} \end{aligned}$$

We want to find out how big we need to make n in order to make this difference be less than ε, and we need to be able to do this for any $\varepsilon > 0$. So, we figure out how big n needs to be so that:

$$\frac{1}{2 \cdot 3^n} < \varepsilon$$

then, in order to get the n by itself, we take the $\log_3$ of both sides:

$$\begin{aligned} -\log_3(2 \cdot 3^n) &< \log_3(\varepsilon) \\ -\log_3(2) - \log_3(3^n) &< \log_3(\varepsilon) \\ -\log_3(2) - n &< \log_3(\varepsilon) \\ -n &< \log_3(2) + \log_3(\varepsilon) \end{aligned}$$

When we multiply both sides by -1, the less than changes into a greater than:

$$n > -\log_3(2) - \log_3(\varepsilon)$$

To recap, what this equation means is that whenever n is greater than the quantity on the right, then the partial sum s_n will be within ε of $1/2$ (the value of the limit). So, for any $\varepsilon > 0$, we can just pick N to be $-\log_3(2) + \log_3(\varepsilon)$, and then this satisfies the condition that whenever $n > N$, the nth partial sum s_n is within ε of $1/2$.

2 *Find an N such that for all $n > N$, the partial sums of $\sum_{n=1}^{\infty} \frac{1}{3^i}$ are within 10^{-6} of $1/2$.*

This problem is the same as the previous one, except with an actual number for ε. We can do the same steps as above and get

$$\begin{aligned} N &= -\log_3(2) - \log_3(10^{-6}) \\ &= -\log_3(2) + 6\log_3(10) \end{aligned}$$

At this point, you might want to use a calculator, but it is also possible to do it by hand. The important thing to remember is that N can be made bigger and it will still work. So, since $3^3 = 27 > 10$, then $\log_3(10) < 3$ and since $3^0 = 1 < 2$, then $\log_3(2) > 0$, so we can choose N as:

$$0 + 6 \cdot 3 = 18$$

3 *Use the definition of a convergent series to show that* $\sum_{n=1}^{\infty}$ *does not equal* 1.

We want to pick a value of ε such that no matter which N someone else chooses, for some value of n, greater than N, s_n differs from 1 by more than ε. Roughly, what it says, is that we want a value of ε such that s_n and 1 differ by more than ε, even if n is very big. As it turns out, we can pick a ε such that s_n and 1 *always* differ by more than ε, but that's not always true in general.

Going through the same steps as in the first problem, we have the following equation for the partial sums:

$$s_n = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

Since $1/3^n$ is always going to be positive, $(1 - 1/3^n)$ will always be less than 1, so s_n will always be less than $1/2$. So, we can pick ε to be anything between 0 and $1/2$, for example $1/4$. We could have also picked 10^{-3} or 10^{-1000}.

In general, the idea with this type of problem would be to find a somewhat simple expression for the partial sums, or at least an inequality. Then to use the inequality to pick the value of ε.