Two heirs to the great chain of being

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Think of the satiric poetry of Jonathan Swift (1667-1745) and the writings of Immanuel Kant (1724-1804) before he achieved fame. How would it be that both contributed to the revolution that shook mathematics in the years 1875-1922 and then —through fractal geometry —reaches diverse natural sciences? The question, at first foolish, improves upon acquaintance —as this sketch hopes to demonstrate. It also turns out to bear most directly on the great mystery of the practical usefulness of mathematics, or —using the grand phrase by Eugene Wigner (1902-1995) —of the unreasonable effectiveness of mathematics in the natural sciences. Stated in the grandest terms, the question deals with the seemingly necessary and unavoidable relations between the purest thought and the most real reality. Thanks to fractal geometry, the mathematical tools of natural science (that is, to use the classic words of Galileo Galilei (1 -) its ‘‘geometrical alphabet),’’ are now being extended in the most unexpected directions. This expansion challenges us to reflect again on the overall nature of the ‘‘foreign relations’’ of mathematics with the various sciences.

A hundred-odd years ago mathematics veered abruptly towards unprecedented and deepening abstraction. For example, it was shown by explicit counterexamples that a continuous function may *fail* to have a derivative. Soon after, geometric constructions whose successive steps were familiar and individually innocuous were combined by Cantor (1845-1918), Peano (1858-1932), and Koch (1870-1924) to generate patterns that seemed on the contrary totally without precedent. A variety of unsatisfactory terms were used to denote them. The terms ‘‘exceptional sets’’ or ‘‘strange sets’’ fitted them well; terms such as ‘‘pathological sets’’ or ‘‘monsters,’’ were less adequate, since deformed or monstrous beings can be capable of life, but the patterns in question were largely lifeless. They were ‘‘disposable’’ or ‘‘throwaway’’ sets: needed once as counterexamples, then never again, except as parts of a barricade mathematicians started to build to separate their field into a pure and an impure component. In one writer’s words, ‘‘the Peano curve cannot possibly be grasped by intuition; it can only be understood by logical analysis.’’ Another prominent writer, a charter member of ‘‘Bourbaki,’’ praised this curve for being ‘‘extravagant’’ and ‘‘a totally non-intuitive monster.’’

However, fractal geometry shows that these very monsters are near inevitable tools in the description of some aspects of Nature that Science had until now neglected. How could this be? Cantor, Peano and their few early followers faced the opposition and scorn of contemporary mathematicians. Most certainly, they did not intend to forge tools for the use of physicists of one hundred years later! More generally, they constructed sets by the systematic use of recursions, wherein each step adds further detail to the background sketched by the preceding steps, Cantor et al. could not possibly have been trying to anticipate the procedure that many areas of physics now find useful in describing the phenomena they call scaling. Yet in fact, Cantor had gone farther than those recent physicists, since they only studied scaling analytically, while Cantor also implemented it in a fully geometric fashion.

This astonishing appearance of pre-adaptation had long baffled me, but now I begin to see a possible historical pattern to it. History cannot explain why these sets succeed in either mathematics or the sciences but it can account for their having been used in both contexts. I could reveal my thesis immediately, but the reader may be amused and interested, as I was, in some ‘‘detective work.’’ Most of the bits of evidence we need are quoted in books of the history of science and the history of ideas and are not new to specialists. But scientists read nothing that is even a few years old and the argument’s interest resides not in individual steps but in the overall chain of reasoning. One could describe it in the spirit of Sherlock Holmes, but it is better to do so in the spirit of Edgar Allan Poe’s *Purloined Letter*.

A basic clue was evident in a few lines that achieved mostly anonymous celebrity:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity (in the molecular sense).

I would like to believe that the author, Lewis Fry Richardson (1881-1953), set out this clue deliberately. He was an extraordinary maverick who is finally acquiring the fame he deserves. In particular, his work on turbulence in the 1920's is now recognized as one of the sources of the theory of physical scaling. It is to illustrate the notion of *turbulent cascade*, that he wrote the four lines in question. It is hard *not* to view them as a parody of a satire written in 1733 by Jonathan Swift (*On Poetry, A Rhapsody*, lines 337-340):

So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey,
And these have smaller Fleas to bit 'em,
And so proceed ad infinitum.

Why, however, should this parody be viewed as a clue, and in what sense could Richardson be a disciple of Swift? What if, in fact, he took the gist of Swift's lines from some later parody, such as this ditty, variously credited either to Augustus De Morgan or to Oliver Wendell Holmes:

Big bugs have little bugs
Upon their backs to bite them.
And little bugs have littler bugs,
And ad infinitum.

In fact, it is not really important whether or not Richardson did a direct take-off of Swift. Either way, he placed himself within a certain old and well-identifiable philosophical tradition. For the present stage of our detective work, it suffices to mention that, as recounted in *The Great Chain of Being* by Arthur D. Lovejoy (Harvard University Press 1936), this tradition harks back to Plato and Aristotle. It was kept alive during the Christian West's Middle Ages by Averroes and his Jewish contemporaries, then revived and flourished, reaching its peak with Leibniz and his eighteenth century followers —including literary types of whom one of the most prominent was precisely Swift! Another was Alexander Pope (*An Essay on Man*, Epistle I, viii):

Vast chain of being! Which from God began;
Natures ethereal, human, angel, man,
Beast, bird, fish, insect, who no eye can see.
No glass can reach; from infinite to thee;
From thee to nothing.

Now we can begin to motivate all these links between eccentric scientists and poets. Moving on to a mathematician in Cantor's mold, let us imagine Helge von Koch building up the snowflake curve for which he remains famous. He starts with a regular triangular ''flea.'' Then he places on the middle of each of its three backs a smaller triangular ''flea,'' then smaller triangular fleas wherever possible on the backs of old or new fleas. And thus, he too, proceeds ad infinitum. Of course this fantasy is not based on any evidence, but it does seem to make my point. Whatever his actual motivation, it is unthinkable that Koch was *not* nourished by the Swiftian currents in our common culture.

A second and different clue pointing one to common roots between the Cantorians and the Richardsonians concerns the study of stellar and galactic clustering. Here is a sensitive topic for those who search for conceptual roots, because professional astronomers are loath to acknowledge any influence from the stargazing riffraff, ''however attractive their conceptions may be in their grandeur'' (to quote the great Simon Newcomb 1835-1909). This disinclination may explain why the first fully described hierarchical model is generally credited to the astronomer Carl Vilhelm Ludvig Charlier (1862-1934). But in fact, he abundantly and explicitly gave credit to a book titled *Two New Worlds* by Edmund Edward Fournier d'Albe (1868-1933; even more of a maverick than L.F. Richardson).

Furthermore, Charlier was greatly interested in and influenced by the early works of Immanuel Kant (1724-1804). Well before his famous *Critiques* books, Kant had written about whether the world is finite or infinite in extent. And on the question of the homogeneity in the distribution of matter, his comments are extraordinarily eloquent and clear-cut, witness these highlights (which should encourage one to savor the fuller text in Munitz 1957):

> I come now to that part of my theory which gives it its greatest charm, by the sublime idea which it presents of the plan of the creation. It is ... natural ... to regard ('the nebulous stars') as being ... systems of many stars ... ('They') are just universes and, so to speak, Milky

> Ways ... It might further be conjectured that these higher universes are not without relation to one another, and that by this mutual relationship they constitute again a still more immense system ... whose termination we do not know, and which perhaps, like the former, is a system inconceivably vast — and yet again but one member in a new combination of numbers! We see the first members of a progressive relationship of worlds and systems; and the first part of this infinite progression enables us already to recognize what must be conjectured of the whole. There is no end but an abyss ... without bounds.

It remains to ponder to what extent this German philosopher, who has influenced astronomers, also influenced mathematicians. It may be that Cantor, of German culture, and Peano, of Italian culture, did not know of Alexander Pope and knew of Jonathan Swift only through Lemuel Gulliver. But each knew the classics well, and was very conscious of historical perspective. In fact, Cantor would have been aghast at being accused of neglecting the thoughts of the earlier great thinkers his ambition was to equal. A major factor that slowed the acceptance of his ideas lay in their deep roots in philosophy and religion. He was schooled in Plato and the scholastics, viewed mathematics as an auxiliary to metaphysics, and wrote far more pages about the former than about the latter. Fortunately, his late mathematical works, which generated the greatest opposition, are beyond our concern, but the same intellectual roots doubtless affected his early work, which concerns us greatly. There can be no doubt of his awareness of the tradition of the Great Chain of Being which ended with Kant.

Based on those two case histories, the reason Cantor et al. and Richardson et al. sound so much alike has ceased to elude me. To heighten the drama, allow me to express my thesis by paraphrasing the last words of Verdi's opera *Trovatore:* these great men had grown in the folds of hostile families, but in intellectual tradition, *they were brothers.*

Of course, this claim concerning history cannot settle the mystery described in the opening paragraph, it merely moves on and changes character: how can it be that the mixture of information, observation and search for introspectively satisfying structure that characterize our sources (men whom physicists like to call ancient scribblers) should repeatedly yield themes so potent that, long after their details have been found to contradict better observation, and the themes themselves have faded away, they continue to provide ways of thought that inspire effective developments in both physics and mathematics?

So far, we needed no specific acquaintance with the actual contents of the conception of the Great Chain of Being, as argued that it was a powerful movement of ideas that could serve as the common fount of two trends which interest us. But digging a bit deeper uncovers an interesting gem. As originally related in the Natural History of Aristotle, the conception of the Great Chain of Being was concerned with the finite versus the infinite, and the discrete versus the continuous. For example, one of its component themes, the so-called principle of continuity, implied that if there is between two given natural species a theoretically possible intermediate type, that type must be realized —and so on ad infinitum. Here we see the origin of the belief in ''missing links'' of all sorts, including chimeras in the original sense this term had in Greek mythology: beasts having a lion's head and a goat's body (and also having the tail of a dragon and spitting fire from their mouths!). As is well known, the belief in biological chimeras was discredited (and even today they are not central to biology). But the suggestive power of ideas is not at the mercy of failure in any particular field! In mathematics, a familiar application consists in the interpolation of the sequence of integers by ratios of integers, then by limits of ratios of integers. Hence, in the Great Chain of Being tradition, every phenomenon defined by a sequence of integers is a candidate for interpolation.

In this light, what about the sequence of dimensions? Since the time of the Pythagoreans, dimension has been associated with integers, but very practical men have started to call out for ''in-between'' shapes that, for example, would participate of both lines and surfaces. Among the seekers one finds students of turbulence, thwarted in efforts to decide whether their process concentrates on ''peas, spaghetti, or lettuce,'' and irritated that different ways of asking the question should yield different answers. Other seekers of the ''in-between'' are found among students of galactic clustering wishing to describe the texture of certain shapes that look streamlike even though they are composed of isolated points. Would it be artificial to proclaim to these sober seekers, who may all be unaware that their concerns are those of ancient scribblings and old Greeks' nightmares, that they are in fact following the well-worn paths towards chimeras? And —switching again for the would-be imitators of Nature to the would-be artists for Art's sake

—could it be merely coincidental that the first actually in-between shapes were first described precisely by our Georg Cantor? The fact became patent in 1919 when Felix Hausdorff (1868-1942) introduced a new nonclassical notion of dimension, and showed that, in the general case, the dimensions of Cantor and Koch monsters *lie between the integers.* (Until age 35, Hausdorff had made a living by philosophy and drama).

Is it not a pity that Leibniz (1646-1716) missed these developments! Yet, actually, he came close. While a Newton could be satisfied with defining derivatives and integrals, and then iterating these operations using an integer index, for a Leibniz they are but a first step. Immediately he thinks of making the order of integer differentiation into a fraction ...

In conclusion, are all these roots real, and do they matter? A jaundiced view of intellectual roots has been expressed by Johann Heinrich Lambert (1728-1777), who thought up essentially the same cosmological system as Kant's, and at about the same time. On becoming aware of this conflict of priority, he expressed the wry comment that it ‘‘will make no impact until an astronomer shall discover something in the sky that could not be explained otherwise; when the system will be found demonstrated a posteriori, the lovers of Greek literature will come and have no rest until they can prove that the whole system had already been known to Philolaus, Anaximander or some other Greek pundit, and that recently it was only rediscovered and embellished. These are the people who find everything in the ancients, provided one tells them what they should look for.’’ Personally, I think that the common roots of the Cantorians and the Richardsonians are real, and that they matter.