

Three Fractal Models in Finance: Discontinuity, Concentration, Risk

BENOIT B. MANDELBROT*

Rather than to attempt to reconstitute the actual lecture given in Siena, this text loosely follows parts of the Preface and Introduction of a book of almost the same title. This book was announced in Siena and is due out at the same time as this paper; it will be referred to as M 1997e (the style of reference is explained at the beginning of the Bibliography).

Introduction

Public opinion and the profession know better than ever before that prices often change with hair-raising swiftness. In 1996, IBM experienced two large near-discontinuities: it fell by more than 10% and later rose by 13.2%. Furthermore, even in the absence of actual jumps, price changes do not occur evenly over time, but tend to be *concentrated* in short “turbulent” periods. Thus, *concentration*, a concept arising from the familiar phenomenon of industrial concentration, generalizes the variation of prices. Wide awareness of this form of concentration is reinforced by failures of portfolios that claimed to be free of *risk*.

My work in finance has been largely devoted to the roles of discontinuity and closely related forms of concentration. Stock Market “chartists”, as contrasted with the “fundamentalists”, believe that charts embody everything needed to predict the future and devise winning strategies. Whatever one’s position on this dispute, I believe that one must understand the structure of charts thoroughly, including features that fail to bring positive returns to the investor. The understanding gained by a thorough exploration is bound to bring significant knowledge about the mechanisms of the financial markets and about the laws of economics. Most importantly, this knowledge is essential for evaluating the unavoidable risks of trading.

The search for winning trading strategies is a goal of every study in

* Yale University, New Haven, CT 06520-8283 USA.

finance, but will not be addressed here. Ultimate scientific explanation is attractive and important, but financial engineering cannot wait for full explanation. That is, it is legitimate to strive towards a second best: a "descriptive phenomenology" that is organized tightly enough to bring a degree of order and understanding.

Indeed, a multitude of consequences that can be tested empirically will be drawn from somewhat abstract statistical algorithms to be described as "scaling". They look simple, even simple-minded, but prove to be "creative", insofar as they generate unexpectedly complicated and structured behavior. This will establish that a wealth of features beloved by chartists *need not* be inserted "by hand", but *may well* follow inevitably from suitable forms of totally random variability.

The word "model" shall *not* denote the mathematical expression of an economic relationship. It will denote a statistical algorithm meant to fulfill an apparently extravagant ambition: it begins with property assumed as "axiom", and is able to produce sample data streams that are hard to distinguish from actual records, either by eye or by algorithm.

At this point in time, my most general model is described by the words "fractional Brownian motion of multifractal time". Without ad-hoc "patch" or "fix", it successfully accounts for many features of the variation of foreign exchange rates, and will be explained below.

All told, my methods of investigation are those of a practicing theoretical and computational physicist. My tools were not to reach physics proper until later, as shall be seen below.

A proper quantitative study of financial markets began in the early 1960s. A reprint of my first contribution to finance, M 1963b {E14}, was included in Cootner 1964, with interesting comments by the editor, Paul H. Cootner (1930-1978). One reads that "Mandelbrot ... has forced us to face up in a substantive way to those uncomfortable empirical observations that there is little doubt most of us have had to sweep under the carpet up to now. With determination and passion he has marshalled, as an integral part of his argument, evidence of a more complicated and much more disturbing view of the economic world than economists have hitherto endorsed. Furthermore, this new view has a strong attraction for many of us".

Elsewhere in Cootner 1964, one reads that "There can be no doubt that Mandelbrot's hypotheses are the most revolutionary development in the theory of speculative prices since Bachelier 1900". This last reference can be viewed as the point of departure of a rational approach to finance, since it was the first to describe the Brownian motion model, which will be discussed below.

Needless to say, my view of the economic world's complexity was not

adopted in 1964, its implications were not faced, and the study of finance continued to rely on the “1900 model” of Bachelier, which pointedly denies discontinuity and concentration. Those obvious defects have become unacceptable.

To tackle discontinuity and concentration, I conceived in the late fifties a tool that has already been mentioned, but deserves elaboration. I concluded that much in economics is *self-affine*; a simpler word is *scaling*. This notion is most important, and also most visual (hence closest to being self-explanatory), in the context of the financial charts. Folklore asserts that “all charts look the same”. For example, to inspect a chart from close to, then from far away, take the whole and diverse pieces of it, and resize each to the same horizontal format known to photographers as “landscape”. Two renormalized charts are never identical, of course, but the folklore asserts that they do not differ in kind. The scholarly term for “resize” is to “renormalize” by performing an “affinity”, which motivated me in 1977 to coin the term “self-affinity”. (Scaling can also be self-similar, as in the bulk of M 1982f {FGN}). The scholarly term for “to look alike” is “to remain statistically invariant by dilation or reduction”.

I took this folklore seriously, and one can say that a good portion of M 1997e studies financial charts as geometric objects.

Brownian motion is scaling also, but it is best to restrict this term to non-Brownian models. Non-Brownian forms of scaling are sometimes called “anomalous”; a better term is “non-Fickian”.

Viewing the renormalized pieces of a chart as statistical samples from the same underlying process, I identified, one after another, several non-Brownian implementations of scaling, and tested them – successfully – in one or another financial context.

In this spirit, the “M 1963 model”, centered on M 1963b {E14}, concerned speculative prices for which long-term dependence is overwhelmed by discontinuities or periods of very fast change.

As soon as advances in computer graphics made it possible, fractal “forgeries” of price records were drawn for the M 1963 model. These forgeries prove to be realistic. This significant discovery became an early exhibit of a surprising and fundamental theme common to all areas of fractal geometry.

In a later “M 1965 model” (M 1965h {H}, M & Van Ness 1968 {H}, and many other chapters in M 1997h, the key factor is long-term dependence.

The “M 1967 model”, described in M & Taylor 1967 {E21}, soon introduced the notion of trading time and pointed out the relevance of a once esoteric mathematical notion called “subordination”.

Finally, M 1972j {N14} introduced the notion of *multifractal*, and

concluded (page 345) by immediately pointing out this notion's possible implications to economics. This is why the term "M 1972 model" will be applied to the approach that I developed in recent years on the basis of multifractals.

The scaling principle of economics incorporates these and all other forms of variability and promises further generalizations. I also used scaling in other contexts of economics, such as the distribution of income and of firm sizes.

M 1963b {E14}, as soon became clear, addressed concerns that were not being faced, that is, answered questions that were not being widely asked, and my reliance on computers was prohibitive. More importantly, the conceptual tools of my work, and its perceived consequences, were resisted. In a way, one may say that I was "victimized" by a case of unexpected historical primacy of financial economics over physics.

An earlier case of such primacy has already been mentioned twice. Odd but true, and implied in the quote from Cootner 1964, a maverick named Louis Bachelier (1870-1946) discovered Brownian motion while studying finance, five years before Albert Einstein and others independently rediscovered and developed it in physics. Eventually, but not until the 1960s, many hands brought Brownian motion back into economics.

Quite similarly, scaling and renormalization were central to my work in finance several years before they were independently discovered and developed in the study of critical collective phenomena of physics, through the work of Fisher, Kadanoff, Widom, Wilson, and others. In fact, "scaling" and "renormalization" are those physicists' terms, and replace my weaker terminology. This "rerun" of the story of Brownian motion confirms that, while economics finds it easy to borrow from established physics, the cases of historical primacy of economics over physics start by being a handicap if they involve overly unfamiliar and untested tools.

Yet another reason made me move out of economics. When P. H. Cootner described M 1963b {E14} as a "revolutionary development", did he think mostly of destruction or reconstruction? The answer is found in Cootner 1964: "Mandelbrot, like Prime Minister Churchill before him, promised us not utopia, but blood, sweat, toil and tears. If he is right, almost all our statistical tools are obsolete..., past econometric work is meaningless... It would seem desirable not only to have more precise (and unambiguous) empirical evidence in favor of Mandelbrot's hypothesis as it stands, but also to have some tests with greater power against alternatives that are less destructive of what we know".

The wish to see improved statistical tests can be applauded without reservation. But it is prudent to fear that "what we know" is not

necessarily the last word. Allow me just one example. When Fast Fourier algorithms became available in around 1964, spectral analysis created great interest among economists. Tests showing some price series to be nearly *white* were interpreted as implying the absence of serial dependence. The result made no sense and was forgotten. But (as reported in Chapter E6 of M 1997e) a promising non-Gaussian model of price variation is white, despite the presence of strong serial dependence! It often seems that novel uses of known statistics may at the same time test a hypothesis and test a test!

In Cootner's already quoted words, my "view of the economic world is more complicated and much more disturbing than economists have hitherto endorsed". This implies that the erratic phenomena to which this book is devoted deserve a special term to denote them, and explain why I chose to call them *wildly random*. By contrast, Brownian motion and most models used in the sciences deserve to be characterized as *mildly random*, and lognormality and some other treacherous forms of randomness that are intermediate between the mild and the wild will be described as *slow*.

The point is this: the specificity of slow or wild randomness in economics could be disregarded for many years, but can be no longer. In particular, new statistical tools are urgently needed.

The diverse obstacles that made my work "premature" in the 1960s have vanished. Computers are everywhere. Physicists and fractalists have developed new modeling tools that can be applied to finance. Abundant financial data are readily available. Events ensure that concern with discontinuity is near-universally shared, and my work is pointedly addressed to the many "anomalies" that bedevil prevailing financial models.

Overall, my ideas now fit in the framework of fractals (hence also of chaos) and of experimentation with new financial products.

1. *An Entirely Pragmatic View of the Slippery Notion of Randomness*

Randomness is an intrinsically difficult idea that seems to clash with powerful facts or intuitions. In physics, it clashes with determinism, and in finance it clashes with instances of clear causality, economic rationality and perhaps even free-will. It is easy to acknowledge that randomness can create its peculiar regularities. But it is difficult to acknowledge that such regularities either could be interesting or could arise in physics or finance. As a result, the fact that any statistical model could be effective seems *a priori* inconceivable and is difficult to acknowledge.

The problem is mitigated in physics because the atoms in a gas are not known individually, and much exaggerated in finance for the opposite

reason. Furthermore, it is difficult in finance to disentangle the roles of the observer and the active participant. Most persons seek financial knowledge for the purpose of benefitting from it, and thereby modifying what they benefit from. But merely describing the markets does not perturb them, and my ambition is simply to observe, describe some degree of order, and thereby gain some degree of understanding. This leads to a pragmatic view described in Chapter 21 of M 1982f {FGN}, titled "Chance as a Tool in Model Making". Several paragraphs of that chapter will now be paraphrased. In short, the notion of randomness is both far more effective and far less assertive than is often assumed or feared.

It is necessary to first comment on the term, *random*. In everyday language, a fair coin is called random, but not a coin that shows *head* more often than *tail*. A coin that keeps a memory of its own record of *heads* and *tails* is viewed as even less random. This mental picture is present in the term *random walk*, especially as used in finance (Section 3).

Of course, statisticians hold a broader view of randomness, which includes coins that are not fair or have a memory. However, more or less explicitly, statisticians ordinarily deal with observations that fluctuate around a "normal state" that represents equilibrium. A good picture of that classical scenario is provided by the edge of a razor blade. When greatly enlarged, it presents many irregularities, but from the user's viewpoint, a high quality blade is practically straight overall, therefore its description splits naturally into a fluctuation and a highly representative "trend". In Chapter E5 of M 1997e, such fluctuations deserve to be called *mild*.

For contrast, consider a coastline like Brittany's or Western Britain's. Taking into account an increasingly long portion will average out the small irregularities, but at the same time inject larger ones. A straight trend is *never* reached and interesting structures exist at every stage.

All too often, however, "to be random" is understood as meaning "to lack any structure or property that would single out one object among other objects of its kind". The question arises, is this a fair characterization of randomness or, on the contrary, is the mathematical notion of chance powerful enough to bring about the strong degree of irregularity and variability encountered in coastlines as well as in financial charts?

The answer to that question came as a surprise: not only is chance powerful enough, but in many cases it goes *beyond* the desired goal such as, for example, the case for this generalized process I introduced in M 1967b {N10} and proposed to call "sporadic". In other words, I perceive a tendency to grossly underestimate the ability of chance to generate extremely striking structures that had not been deliberately inserted in advance.

However, fulfilling this goal demands forms of randomness that are far broader than is acceptable in the bulk of statistics. Once again, the physicists' concept of randomness is shaped by mild chance that is essential at the microscopic level, while at the macroscopic level it is insignificant. In the scaling random fractals that concern us, on the contrary, the importance of chance remains constant over a wide range of levels, including the macroscopic one. This non-averaging change is described in Chapter E5 of M 1997e as *wild*; I continually explore it, and claim that it, and *not* mild chance, is the proper tool in finance and economics.

Be that as it may, the relationship between unpredictability and determinism raises fascinating questions, but this work has little to say about them. It makes the expression "at random" revert to the intuitive connotation it had at the time when it entered medieval English. The original French phrase "un cheval à randon" is reputed to have been unconcerned with underlying causes, such as the horse's psychology, and merely served to denote an irregular motion the horseman could not fully *predict* and *control*.

Thus, while chance evokes all kinds of quasi-metaphysical anxieties, I am little concerned with whether or not Einstein's words, "the Lord does not play with dice", are relevant to finance. I am also little concerned with mathematical axiomatics. The reason I make use of the theory of probability is because *there is no alternative*: it is the only mathematical tool available to help map the unknown and the uncontrollable. It is fortunate that this tool, while tricky, is extraordinarily convenient and proves powerful enough to go well beyond mild randomness to a wild and "creative" state of that concept.

Let us draw some other consequences from the preceding combination of a *credo* and a promise. To be entitled to use probability theory, there is *no need* to postulate that *every* financial and economic event is generated by chance, rather than by cause. *After the fact (ex-post)*, one may find uncontroversial or reasonable causes for at least some features. But *before the fact (ex-ante)*, the situation is very different. In order to move towards a quantitative approach to economics (a "rational economics" to echo the grand old term "rational mechanics"), one must *unavoidably* resort to probability theory. Further subtle but significant differences between *ex-post* and *ex-ante* are found throughout my work.

To believe that the concept of randomness has far more power than it is credited with is essential, but is not enough. One must construct actual random processes that use few inputs, are tightly organized and fit the data so well as to yield a degree of understanding. To show this is feasible is an ambition of my work.

2. Graphics, the Computer, Statistics and Beyond: "Index Numbers" and Other Summaries of the Data

"To see is to believe", and to prove that the last paragraph of the preceding section is valid, the easiest and most convincing path consists in allowing the eye to compare the actual data to the outputs of random processes, without the technical details given in Chapter E6 of M 1997e. The reader is, therefore, encouraged to compare Figures 1 and 2. The latter is a simulation of a surprisingly simple process (which took a

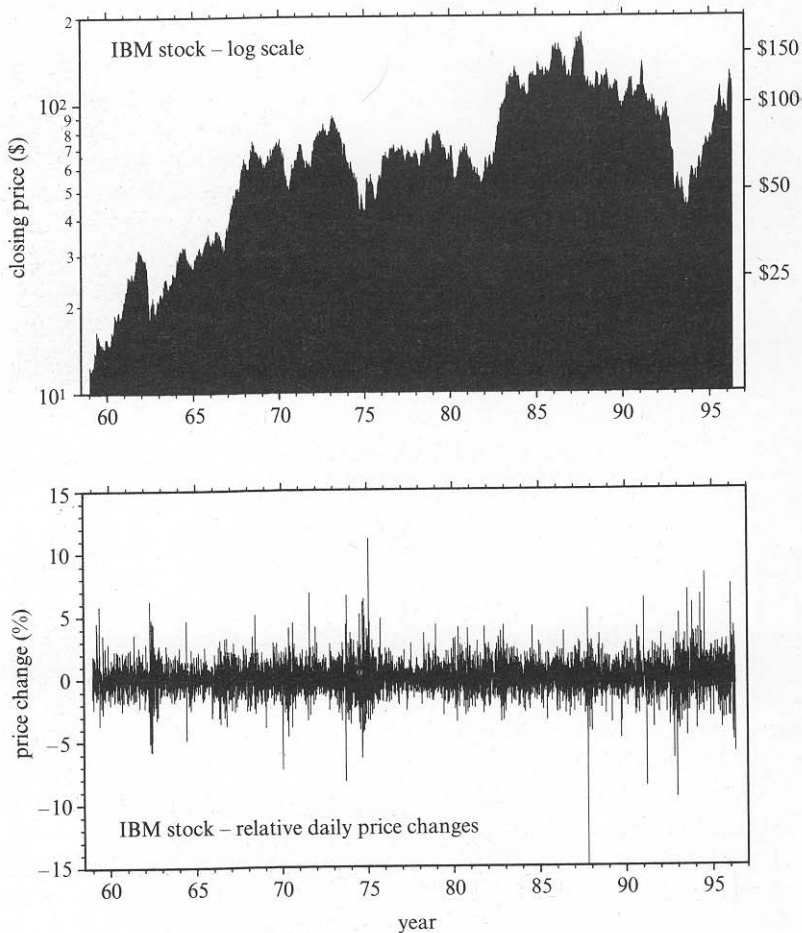


Figure 1

surprisingly long time to be identified!) and it can be “tuned” to achieve a wide range of different behaviors.

My claim is that ability to imitate is a form of understanding. To preempt a challenge, imagine that a statistical test proclaims that the data in Figure 1 are actually very different from the simulation in Figure 2. If so, what should the proper response be: to dismiss the evidence of the eye or to look carefully for hidden assumptions that may have biased the

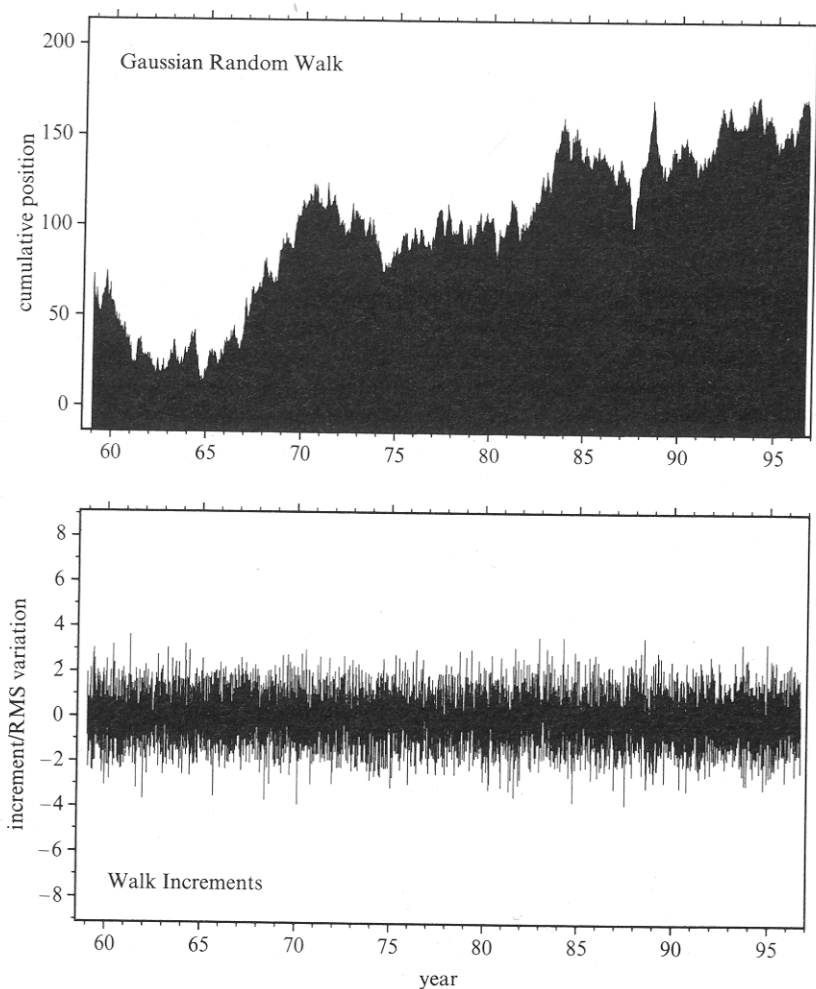


Figure 2

statistical test? This second possibility is the one that this section proposes to examine in detail. Many readers may want to proceed to Section 3.

Throughout most of history, computation and pictorial representation were at best prohibitively expensive and mostly impractical or even inconceivable. Deep philosophical reasons were put forward to justify the fact that the eye became suspect and was almost completely banished from "hard science". In my opinion, those philosophical reasons were unconvincing, and the real reasons for forsaking the eye was simply practical. Therefore, the advent of cheap computing and graphics will have a profound and increasing impact throughout the sciences. Of special concern to us is its impact on probability theory and statistictive fields are related but separate and best examined in turn.

In the past, random processes could only be investigated through formulas and theorems. The resulting knowledge is invaluable, but three episodes made me conclude, long ago, that it is incomplete.

The shallowness of my earlier understanding of random walk first became obvious after I spent hours dreaming while examining, again and again, the sole illustration in Feller 1950 (Volume 1). This illustration, a record of painstaking actual throws of a coin, is reproduced, with comments, in Figure 4 of M 1963e {E3}, and also in Plate 241 of M 1982f {FGN}. William Feller later confided to me that this was one of several possible illustrations prepared for him, and was chosen because the alternatives were even further removed from the readers' prejudices. It was disappointing that a figure I found inspiring should exemplify the fact known to everyone, that Man often uses pictures to *disguise* reality, instead of illustrating it. But Man also uses words for the same purpose! Be that as it may, I think that the publication of *all* of Feller's pictures would have provided clearer "intuitive" or "in the fingers" understanding.

Second episode: following immediately upon M 1963b {E14}, Berger & M 1963 {N} faced the challenge of making it obvious that a certain physical phenomenon involved a degree of randomness well beyond the mild. To force conviction, they had to resort to a hand-cut wire model.

Third episode: as soon as crude "Calcomp" tracing tables could be attached to a computer, M & Wallis 1969 a, b, c {H} hastened to put them to use in illustrating a process that will be discussed below. Comparing the data with the sample functions of the M 1965 model and other models, we saw instantly that certain models could not possibly be correct, while other models seemed adequate. The "objective" statistical tests available at that time provided less clear-cut distinctions, confirming that they had been devised to deal with a context substantially different from the context of fractals. Only half in jest, we thought that the

calculations involving existing statistical techniques were not only a way to test a model, but also to test the tests.

Given the minimal cost of sample graphs, I never felt we erred by producing too many, but often fear we err by producing too few.

The same issue can be seen under another light. Under the old technological constraints, the long lists of data provided by observation and measurement could not be graphed or manipulated usefully, and it was imperative to begin by compressing them drastically. In economics, of course, this compression yielded diverse “indicators” or “index numbers”. The most classical and simplest index numbers are (weighted) averages, but it is useful to use the same term more widely, in particular, for moments of order higher than 1.

Needless to say, the proper selection of index numbers is an endless source of controversy, and skeptics claim that suitable weights can yield any result one wishes. Nevertheless, it is widely believed that moments are an intrinsic concept. But are they really? In mechanics, it is indeed true that the first and second moments, weighted by mass, yield a body’s center of gravity and radius of gyration. In finance, the value of a portfolio is a non-controversial first sample moment; more controversial is “cost-of-living”. But, *ex-ante*, second and higher moments are far from unquestionable. It is true that they play a central role in Taylor series (to be touched upon at the end of Chapter E5 of M 1997e), but their main role is to provide quantitative “index numbers”, whose usefulness must *not* be viewed as obvious but must instead be established separately in each case.

An important point is that, *ex-post*, the compression implicit in the moments is not necessarily useful but depends sharply on specific circumstances to be distinguished in Chapter E5 of 1997e. In the cases of *mild* randomness, as exemplified above all by independent Gaussian variables, very simple compression is “sufficient” and preserves the important information. However, much of M 1997e argues that economics and finance (as well as many fields of natural science) are characterized by forms of randomness that are *not mild at all*. A prime characteristic of *wild* randomness is that familiar index numbers altogether cease to be representative in their case. What should be done until new and more appropriate statistical tools become available? I think that one will have to live with graphical tests, and learn to perform them with care and without haste, taking full advantage of computer graphics.

“*When exactitude is elusive, it is better to be approximately right than certifiably wrong*”. To be unquestionably correct is a nice idea but is not an option, and the Stock Market wisdom quoted in this paragraph’s title is an excellent characterization of one aspect of my approach. In particular, the Brownian model of Section 3 claims to be valid without

restrictions, which is certainly quite wrong, while the scaling models of Sections 6 to 8 only claim to be approximately exact over a limited range of applicability. That is, when the data are compared with my theoretical distributions, one should expect systematic errors of specification that are *larger* than errors due to statistical fluctuations.

More detailed reasons for tolerating the imperfection described in this subsection's title will be discussed in Section 5 and in Chapter E2 of M 1997e. A discussion of this "necessary tolerance" should be part of statistics, but is not.

Even the best objective statistical tests are not of universal validity. Who is testing the testers? My attitude is deeply colored by publications that reexamined my M 1963 model quantitatively and concluded that "objective" tests contradict the "subjective" claims I based on graphical evidence. Actually, that graphical evidence was confirmed again and again, while the objective tests are forgotten, as they deserved to be. Indeed, there is no theorem without assumption, and even the best statistical test can only be used under certain conditions of validity.

Take spectral analysis: this is unquestionably a powerful tool, but a very tricky one: Chapter E6 of M 1997e (Section 3.5) will describe a significant "blind spot" of spectral whiteness that concerns non-Gaussianity. It is not a matter of mathematical nitpicking, but it directly concerns finance. More generally, if my models are in fact close to being correct, reality lies beyond the domain of applicability of many universally accepted statistical tests, and we should expect to find that these tests will conclude against my models' validity. All too often, strange as it may sound, the conclusions yielded by such statistical criteria evaluate both the model and the test in some inextricable combination, from which little of use can be inferred.

Once again, while statistics works out the above challenges, we have no choice but to rely on graphics. It is worth noting that fully fleshed-out and detailed pictures – not skeletal diagrams – put *no* premium on concision, therefore on compression. But they put a heavy premium on the ability of the eye to recognize patterns that existing analytic techniques were not designed to identify or assess.

3. *"Random Walk Down the Street", Mild Randomness, Brownian Motion (the "1900" Model), and Martingales*

Scaling models are meant to replace the simplest model of price variation, which Malkiel 1973 breezily called "Random walk down the street". Every version assumes that prices change randomly and each price change is *statistically independent* of all past ones. The probabilists'

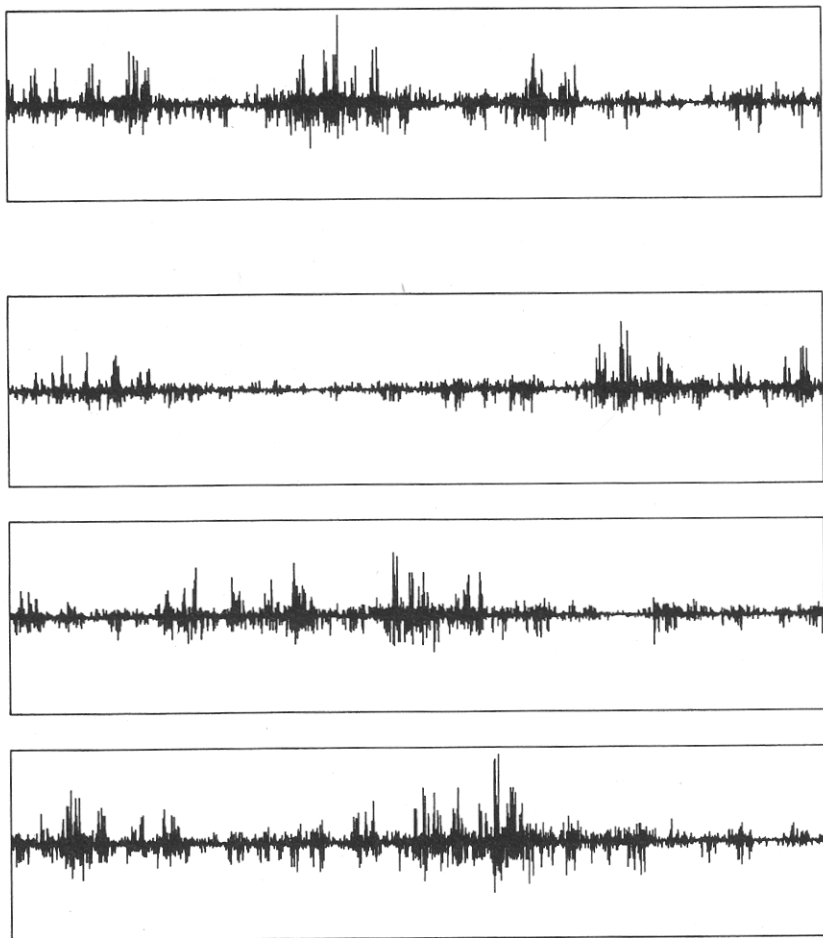


Figure 3

original random walk proceeds in equal steps, up or down, equally spaced in time. In Figure 3, the steps are so small as to be indistinct.

Another basic version assumes that price changes follow the Gaussian (“bell curve”) distribution, which allows for a “mild” level of scatter. Typical generalizations assume or imply that individual price changes need not be Gaussian, but are only mildly scattered. In M 1997e, the technical meaning of the term “mild” is sketched in Chapter E2 and described in Chapter E5 and E6. Quite appropriately, those examples interpret the word *walk* to denote a motion that proceeds in *steps*, while the alternative M 1963 model proceeds in *jumps*.

3.1. The "Ordinary" Wiener Brownian Motion

The continuous-time counterpart of random walk advanced in Bachelier 1900, is now called *Brownian motion*, and will be denoted as $B(t)$. Bachelier's discovery of Brownian motion in financial speculation occurred years before physicists discovered it in the motion of small particles, and decades before a mathematical theory of $B(t)$ was provided by Norbert Wiener. The tale is recounted in M 1982f {FGN}, p. 392. Hence, the composite term *Wiener Brownian motion* (WBM) will be used in case of ambiguity, in particular, when one must provide contrast with *fractional* Brownian motion (Section 7); a term used on occasion is "B 1900 model".

The main properties of Wiener Brownian motion are best listed in two categories, as follows.

3.1.1. Invariance Properties of Wiener Brownian Motion

Invariances are familiar in the hard sciences. Thus, classical geometry begins by investigating what can be done when the shapes one deals with reduce to lines, planes, or spaces. And the simplest physics arises when some quantity such as density, temperature, pressure, or velocity is distributed in homogeneous manner. The line, plane or space, and the homogeneous distribution on them, are *invariant* under both *displacement* and *change of scale*; in technical terms, they are both *stationary* and *scaling*.

Both properties extend to Wiener Brownian motion.

- *Statistical stationarity of price increments.* Equal parts of a straight line can be precisely superimposed on each other, but this is not possible for the parts of a random process. However, samples of Wiener Brownian motion taken over equal time increments can be superimposed in a statistical sense.

- *Scaling of price.* Moreover, parts of a sample of Wiener Brownian motion corresponding to non-overlapping time increments of *different* durations can be suitably rescaled so they too can be superimposed in a statistical sense. This key property implements scaling: except for amplitude and rate of change, the rules of higher- and lower-frequency variation are the same as the rules of mid-speed frequency variation.

3.2. More Specialized Properties of Wiener Brownian Motion

Stationarity and scaling do not suffice to determine Brownian motion. It also has the following properties.

- *Independence of price increments.* Knowing the past brings no knowledge about the future.

- *Continuity of price variation.* A sample of Brownian motion is a continuous curve, even though it has no derivative anywhere. (A technicality deserves to be mentioned once: the above properties only hold almost surely and almost everywhere).

- *Rough evenness of price changes.* The eye and the ear are more sensitive to records of changes than of actual values. A record of Wiener Brownian price changes, over equal time increments Δt , is a sequence of independent Gaussian variables. A telling term for this process is “white noise”. The ear hears it like the hum on a low-fidelity radio not tuned to any station. The eye sees it as a kind of evenly spread “grass” that sticks out nowhere. A telling sample is shown in the bottom of Figure 2.

- *Absence of clustering in the time locations of the large changes.*
- *Absence of cyclic behavior.*

3.3. The Martingale Assumption, Pro or Con

Another basic property $B(t)$ must be discussed separately. Bachelier 1900 introduced $B(t)$ as the easiest example he knew of a far broader class of processes now called *martingales*, which embody the notion of “efficient market” and successful arbitraging.

Prices are said to follow a martingale if they somehow acquire the following desirable property: whether the past is known in full, in part, or not at all, price changes over all future time spans have an expectation equal to 0.

This definition allows properties other than the expectation to depend upon the past (as they do in M 1966b E19}). The notion of martingale was a bold hypothesis and a major breakthrough in 1900 and eventually received wide notice, witnessed by Cootner 1964. It does remain attractive and enlightening, but raises serious difficulties.

A first difficulty is this. A positive martingale always converges; that is, it eventually settles down and ceases to vary randomly (Samuelson 1965). Conversely, a martingale that continues to vary randomly must eventually become negative. For example, a random walk eventually becomes negative. True, a “proportional effect” argument makes it customary to postulate that it is the *logarithm of price* that is Brownian, or at least a martingale. When this is the case, price itself cannot become negative, but ceases to be a martingale. Therefore, the “efficient market” justification for martingales disappears.

While a martingale implements the ideal of an efficient market, is it possible to implement it by arbitraging, even under ideal conditions? The answer is *yes* under the conditions postulated in M 1966b {E19}. But the answer is *no* in M 1971e E20}, which postulates that non-arbitrated price

follows fractional Brownian motion, a generalization of $B(t)$ to be discussed in Section 7. This function is not a martingale, and *cannot* be arbitrated to become one.

4. *Brownian Motion's Inadequacies as Model of Price Variation, and Sketch of Proposed Replacements*

Brownian motion is far and way more manageable than any alternative. An immense mathematical literature grew around it, and recently developed "financial mathematics" draws extremely long mathematical inferences from the assumptions that prices follow a martingale, and/or that $B(t)$ applies very exactly to prices. Unfortunately, $B(t)$ is an extremely poor approximation to financial reality. Soon after 1900, Bachelier himself saw that the data are non-Gaussian and statistically dependent. Thus when the model in Bachelier 1900 was "discovered" and confronted with reality, those discrepancies were independently observed by many authors. Thus, Osborne 1962 describes trading as tending to come in "bursts" and Alexander 1964 concluded that price variation is non-stationary. In the editor's comments of Cootner 1964, p. 193, one finds the suggestion that the bursts may be linked to the model of Berger & M 1963; not surprisingly, I had the same idea, found it difficult to implement, but implemented it in due time – as will be seen in Section 8.

More recently, of the many authors who attempted to challenge my models, *every one* reports that $B(t)$ is convenient, but *not one* reports that price variation is, after all, Brownian.

4.1. *A List of Discrepancies Between Brownian Motion and the Facts*

- *Apparent non-stationarity of the underlying rules.* Diagram P in Figure 1 is an actual record of prices. Different pieces look dissimilar to such an extent that one is tempted *not* to credit them to a generating process that remains constant in time. While a record of Brownian motion changes looks like a kind of "grass", a record of actual price changes (diagram I of Figure 1) looks like an irregular alternation of quiet periods and bursts of volatility that stand out from the grass.

- Repeated instances of *discontinuous change*. On records of price changes, discontinuities appear as sharp "peaks" rising from the "grass".

- Clear-cut *concentration*. A significant proportion of overall change occurs within clear-cut periods of high price variability. That is, the "peaks" rising from the "grass" are not isolated, but bunched together.

- Conspicuously *cyclic* (but not periodic) *behavior*. For example, the real price series shown in Figure 1 shows conspicuous "cycles".

It will be seen that the preceding discrepancies can be traced to two characteristics of a more theoretical nature.

- *The long-tailed ("leptokurtic") character of the distribution of price changes.* An especially sharp numerical test of the instability of the sample variance is provided by the analysis of cotton data in Figure 1 of M 1967j {E15}: over 50 sub-samples of 30 days, the sample variance ranged a hundred-fold.

- *The existence of long-term dependence.*

4.2. *The Noah and Joseph Effect, Taken Singly or in Combination*

Those and other flaws of Brownian motion are widely acknowledged, but the usual response is to disregard them or to "fix" WBM piecemeal, here and there, as needed. The resulting "patchwork" is discussed in Section 4 of Chapter E2 of M1997e. My approach is very different. Instead of seeking a grand "model of everything", I moved in successive piecemeal steps, adding generality and versatility as suitable tools became available.

This strategy began by tackling non-Gaussian tails and long dependence separately. Reflecting two stories in the Bible, those of the Flood and of the Seven Fat and Seven Lean Cows, the underlying phenomena were called, respectively, *Noah* and *Joseph Effects* (M & Wallis 1968 {H}). The M 1963 model (Section 6) concerns cases where serial dependence is unquestionable, but a stronger driving feature resides in non-Gaussianity. The M 1965 model (Section 7) concerns cases where non-Gaussianity is unquestionable, but the strongest driving feature resides in serial dependence.

Because of their simplicity, the M 1963 and M 1965 models remain instructive and essential, but they are obviously oversimplified. The M 1967 model simply rephrases the symmetric case of the M 1963 model, and the M 1972 model goes further and tackles non-Gaussianity and serial dependence simultaneously. The models in Sections 6 and 7 generalize Brownian motion in two directions one may call orthogonal to each other, and Section 8 brings those two generalizations together again, as special cases of the M 1972 model. Ways to separate Noah and Joseph features in a record of real data are tackled at the end of Section 7.

4.3. *Is it Fruitful to Emphasize "The Exceptional", Even at the Cost of Temporary and Comparative Neglect of "The Typical"?*

The distinction between the typical and the exceptional is ancient, and my stress on discontinuity and concentration has been criticized. Clearly,

when faced with rare events, Man finds it difficult to avoid oscillating between overestimation and neglect.

Most common is a stress on the typical. It motivated Quételet to his concept of "average man", and we read the following in the Preface of a famous treatise, Marshall 1890. "Those manifestations of nature which occur most frequently, and are so orderly that they can be closely watched and narrowly studied, are the basis of economic as of most other scientific work; while those which are spasmodic, infrequent, and difficult of observation, are commonly reserved for special examination at a later stage: and the motto *Natura non facit saltum* is specially appropriate to a volume on Economic Foundations... [T]he normal cost of production... can be estimated with reference to 'a representative firm'..."

At first, these words seem to contradict an opinion expressed by Jacques Hadamard, that "it is the exceptional phenomena which are likely to explain the usual ones". But the case of a very concentrated industry suggests that the two viewpoints need not be in contradiction. Many believe, as I do, that emphasis on the largest firms agrees with Hadamard's opinion when "exceptional" is interpreted as meaning "concerning few entries in a list of all firms". But it agrees with Marshall's when "representative" is interpreted as meaning "concerning a large proportion of persons on the list of all the employees of those firms".

5. Invariance Principles: Stationarity and Scaling

The fractal approach to finance and economics rests on two features.

One is a profound faith in the importance of invariances and in the possibility of identifying stationarity and scaling as invariance principles in economics. This will be elaborated upon in Chapter E2 of M 1997e.

The second feature is the recognition that probability theory is more versatile than generally believed, and the willingness to face several distinct "states of randomness". When suitably chosen, a scaling random process can allow variation that will be described as "wild" in Chapter E5 of M 1997e. The sample functions of wild processes contain significant features that were *not* deliberately incorporated into the input, yet, without a special "fix", achieve one or several of the following properties.

- *Repeated instances of sharp discontinuity* can combine with *continuity*. The fact that non-Brownian scaling random processes can be discontinuous and concentrated is extraordinarily fortunate; it is not a mathematical pathology that could be source of concern.

- *Concentration* can automatically and unavoidably replace *evenness*.

- *Non-periodic cycles* can automatically and unavoidably follow from long-range statistical dependence.

5.1. Principles of Invariance

In mathematics and physics, such principles are a staple and the key to a wonderful wealth of consequences drawn from one simple idea. But they are not an established part of economics. I recall the eloquence of Jacob Marashak (1898-1977), when proclaiming that the single economic invariance he could imagine concerned the equality between the numbers of left and right shoes, ... and even that could not be trusted. Marashak was doubtless thinking of the basic invariance of mathematics and theoretical physics, which are stated as absolute. However, as already mentioned in Section 2, other parts of physics find it extraordinarily useful to work with invariances that are approximate and have a limited range of applicability.

Thus, I propose to abandon Wiener Brownian motion as a model, but endeavor to preserve stationarity and scaling as basic invariance principles of economics.

5.2. Scaling Under an Especially Critical Form of Conditioning

The probabilists' notation. This notation represents random elements by capital letters, and their actual values by corresponding lower case letters. $\Pr \{\text{"event"}\}$ will denote the probability of the "event" described between the braces. EX will denote the expected value of the random element X . Furthermore, words like *scaling*, *Gaussian*, *lognormal* will be used as substantives, to mean scaling, Gaussian or lognormal distributions.

The scaling distribution. As applied to a positive random variable, the term *scaling*, is short for *scaling under conditioning*. To condition a random variable U specified by the tail distribution $P(u) = \Pr \{U > u\}$, suppose that it becomes known that U is at least equal to w . This knowledge changes the original unconditioned U to a conditioned random variable W . Using a vertical slash to denote conditioning, the tail distribution of W is

$$P_w(u) = \Pr \{W > u\} = \Pr \{U > u \mid U > w\} = \frac{P(u)}{P(w)}$$

Now take the tail distribution $P(u) = Cu^{-\alpha} = (u/\tilde{u})^{-\alpha}$. When $w > \tilde{u}$, conditioning yields $P_w(u) = (u/w)^{-\alpha}$. This expression is functionally identical to $P(u)$. The sole response to conditioning is that the scale changes from \tilde{u} to w . Hence, the tail distribution $P(u) = Cu^{-\alpha}$ is denoted by the term *scaling*. In this wording, "Pareto's law" is an empirically

established finding that states that “the frequency distribution of personal income is scaling”. Conversely, $P(u) = Cu^{-\alpha}$ is the only distribution that is scaling under this particular conditioning.

By the logarithmic transformation $V = \log_e U$, this invariance property reduces to a well-known invariance property of the exponential distribution $\Pr\{V > v\} = \exp[-\alpha(v - \tilde{v})]$. Conditioned by $V > w > \tilde{v}$, the tail distribution becomes $P_w(v) = \Pr\{V > v \mid V > w\} = \exp[-\alpha(v - w)]$, which is identical to $\Pr\{V > v\}$, except for a change of *location* rather than *scale*.

Starting from an exponentially distributed V , a scaling U is obtained as $U = \exp V$. The logarithmic transformation is simple, the exponential is well-known, and the passage from V to U is obvious. Therefore, one may presume that no conclusion that is at the same time new *and* interesting can be obtained concerning the scaling U . The interesting surprise is that this presumption is totally unwarranted. The transformation from U to V raises new and difficult questions that go beyond technical detail to deep and concretely relevant issues. Therefore, the scaling property that follows from $P(u) = (u/\tilde{u})^{-\alpha}$ has far-reaching consequences. M 1997e is largely devoted to studying them, yet makes no claim of coming close to exhausting the topic.

Experimental measurement of the scaling exponent, and practical lack of meaning of high values of α . The exponent α is typically measured on the straight portion of a graph of $\alpha \log \Pr(U > u)$ versus $\log u$. The discussion that accompanies Figure 1 in M 1963e {E5} underlines that experimental work should give little or no credence to high values of α . Such values are entirely determined by observations for which the range of values of $\log u$ is small, making it harder to ascertain the straightness of doubly logarithmic graphs, and errors in α are far larger when α is large than when α is small.

5.3. *Wild Randomness and its Surprising Creativity*

Stationary and scaling suffice to derive many facts from few assumptions. *Ex-ante*, as already observed, they seem simple-minded. *Ex-post*, they turn out to be surprisingly “predictive” or “creative”, in a sense developed in Section 10 of Chapter E1 of M 1997e and elsewhere in this book. Implemented properly and helped by the evidence of computer graphics, they suffice to account for an extraordinary wealth of complicated behaviors, all bound together in what will be described as a “tightly organized phenomenology”. The surprising possibility of such organization is essential from the viewpoint of “understanding”, and Section 10 of Chapter 1 of M 1997e will argue that it is a second best to

full explanation. This “creativity” is the second feature underlying the fractal approach to finance, and expresses a mathematical possibility that is central to every aspect of fractal geometry.

More specifically, the mathematical concept of *stationary randomness* is far less restrictive than generally believed. Properly turned, it generates structures whose richness is well beyond the power of Brownian and near-Brownian randomness, which will be described as “mild”. That is, stationary and scaling processes also extend to the very different forms of randomness that will deserve to be described by the provocative term, “wild”. What we shall see is that many of the observed facts that motivate other writers to propose diverse “fixes” to Brownian motion (see Section 4 of Chapter E2 of M 1997e) can also be accounted for by suitable forms of wild randomness.

6. *Fractals in Finance, Stage I: the “M 1963” Model For Tail-driven Variability and the “Noah Effect”*

The M 1963 model assumes that successive price changes are independent and highly non-Gaussian but stationary and scaling. In practice, it adequately addresses price records in which the long-tailedness of the changes is dominant, and their serial dependence can be studied as a later and closer approximation. This situation turned out to be a good approximation for commodity prices and other examples examined in M 1963b {E14} and M 1967j {E15}.

Given a price series $Z(t)$, write $L(t, T) = \log_e Z(t+T) - \log_e Z(t)$. The M 1963 model assumes that $L(t, T)$ follows a probability distribution called *L-stable*. When successive $L(t, T)$ are independent, $\log_e Z(t)$ is said to follow a random process called *L-stable motion* (“LSM”). The significant parameter is an exponent α ; its range could be $[0, 2]$, but in the case of price changes, it narrows down to $[1, 2]$. Wiener Brownian motion is the very atypical limit case of L-stability for $\alpha = 2$.

The limitation to $\alpha < 2$ is a significant irritant. It makes L-stability inappropriate for certain prices, and perhaps also for certain forms of income investigated in M 1963i {Appendix IV to E10}. Section 8 will show how a generalized model extends the range of α beyond 2.

6.1. *The Original Evidence for the M 1963 Model: the Case of Cotton*

Figure 4, which provided the earliest evidence, first appeared in M 1962c, and was promptly reproduced with detailed explanations in M 1963b {E14}, then in many references including p. 340 of M 1982f {FGN}. This original empirical test of L-stability used Pareto-style log-log

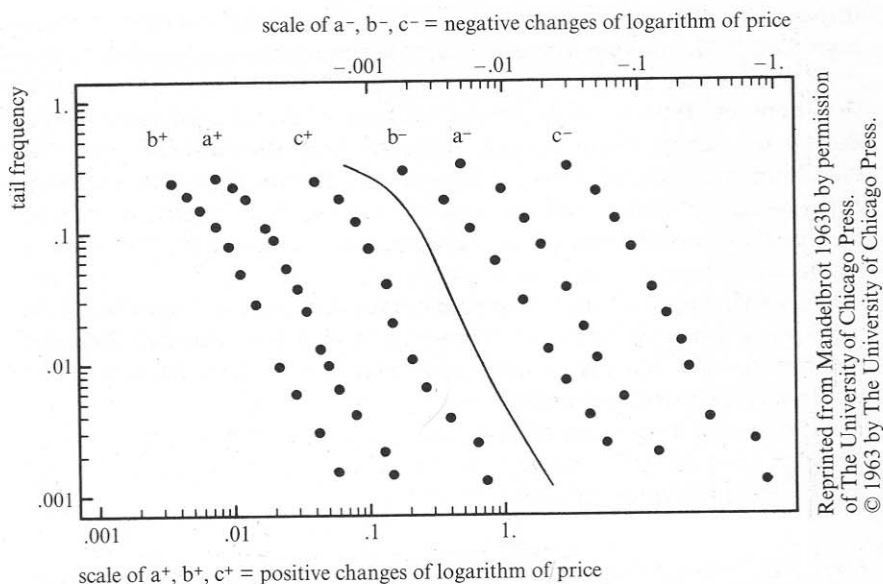


Figure 4

plots. Here is a description translated in slight paraphrase from Mandelbrot 1962c.

“Denote by $Z(t)$ the spot price of cotton, namely, the price for immediate delivery on day t .

“Curves a^+ and a^- represent, for the period 1900-1904, the empirical frequencies $\text{Fr} \{L(t, T = \text{one day}) > u\}$ and $\text{Fr} \{L(t, T = \text{one day}) < -u\}$.

“Curves b^+ and b^- represent, for the period 1944-1958, the empirical frequencies $\text{Fr} \{L(t, T = \text{one day}) > u\}$ and $\text{Fr} \{L(t, T = \text{one day}) < -u\}$.

“Curves c^+ and c^- represent, for the period 1880-1940, the empirical frequencies $\text{Fr} \{L(t, T = \text{one day}) > u\}$ and $\text{Fr} \{L(t, T = \text{one day}) < -u\}$.

“Both coordinates are logarithmic for all $n \times$ curves. To my knowledge, the evidence concerning price variation has never been presented in this way.

“Those various curves quickly become straight lines having the same slope of approximately $\alpha = 1.7$. Therefore, we can write

$$\log [\text{Fr} \{L(t, T) > u\}] \sim -\alpha \log u + \log C'(T),$$

$$\log [\text{Fr} \{L(t, T) < -u\}] \sim -\alpha \log u + \log C''(T),$$

“Thus Figure 4 suggests that the tails are asymptotically ruled by the scaling distribution (see Section 5.2), with the same α exponent

throughout. We also observe that $C' \neq C''$, which reveals a slight asymmetry. The average value of $L(t, T)$ is practically zero.

"We see that a^+ is parallel to b^+ , and a^- is parallel to b^- . This shows that between 1904 to 1958 the distribution of $L(t, 1)$ did not change, except for scale. There is also evidence (not shown here) that the distribution of $L(t, 1)$ changed little from 1816 to 1940. The fact that curves a^+ and c^+ and a^- and c^- are parallel shows that the distributions of $L(t, T = \text{one month})$ and of $L(t, T = \text{one day})$ are identical, except for a change of scale.

"In a first approximation, the six curves displayed in Figure 4 can be superposed on each other by horizontal translation, showing that the distribution of $L(t, T)$ is *L-stable under change of T*. This feature will be interpreted as a strong quantitative symptom of scaling".

Deviations from exact superposition are full of meaning, as shown in Chapter 14 of M 1997e, both in the text which reproduces M 1963b, and in Appendix III which reproduces M 1972b.

6.2. *A Deviation from Invariance can be Significant Statistically, Without being Significant Scientifically*

As mentioned in Section 2, careful testing will doubtless show that prices exhibit statistically significant deviations from stationarity and scaling.

To elaborate, each scaling model of price variation claims to describe properties that apply (up to size factors) to effects at short, middle or longish time scales. Moreover, the M 1963 model involves one basic parameter, α . Specific properties such as a probability of a given kind of ruin can be evaluated, as seen in Section 3.2 of Chapter E6, and the results can be affected dramatically by the value of α . Thus, in the range around $\alpha = 1.7$, the probability of ruin may be *approximately* 10^{-1} , while in the (Brownian) limit case $\alpha = 2$, it may be *exactly* 10^{-20} . Now, what about the actual α ? By visual inspection, the cotton prices yielded $\alpha = 1.7$, but one must expect that short, medium and longish time spans will yield quantitative estimates of α that slightly differ from each other. Hence, the above-mentioned probability of ruin may, in fact, differ from 10^{-1} by a factor ranging between 1/2 and 2, or perhaps even between 1/3 and 3.

My prudent vagueness about the value of α was criticized by P. H. Cootner. I reported those reservations to William S. Morris, who had no difficulty convincing me that the resulting uncertainty about the probability of ruin pales into insignificance. There is a far greater difference between the uncertain value of "approximately 10^{-1} " relative to

$\alpha \sim 1.7$, and the certainly incorrect value of “exactly 10^{-20} ” relative to $\alpha=2$. It is fair to criticize the M 1963 model for being insufficiently precise, but only after praising it for providing a correct order of magnitude.

6.3. *Beyond the M 1963 Model*

Be that as it may, the M 1963 model was pointedly only meant to account for *certain* prices. Given the messiness of the data, it would be reckless not to fear that strict invariances are never encountered. But my systematic policy is to first seek improved models that preserve and generalize stationarity and scaling.

Sections 7 and 8 hope to convince the reader that this has been a fruitful strategy. But Chapter E2 of M 1997e argues that a sensible person should expect some features of the markets to contradict scaling. That is, *ad-hoc* “fixes” or “touch-ups” may eventually become necessary. But, as argued in Section 4 of Chapter E2 of M 1997e, those fixes must not be applied to Brownian motion, but instead to a model that is already part way to the truth.

7. *Fractals in Finance; Stage II: the “M 1965” Model for Dependence-driven Variability and the “Joseph Effect”*

M 1963b {E14}, where the M 1963 model was first described, specifically acknowledges the existence of serial dependence in price changes, but the model itself approximated by postulating independence. This attitude was criticized for many reasons. In particular, every form of so-called “static” description is viewed as less desirable than a “dynamical” one that promises to be a possible basis for both portfolio management and conceptual understanding. I do not share this scorn for statics, but moved beyond the M 1963 model in several steps. In a first step, M 1965h {H} addressed records in which change is dominated by global (long-run) dependence and the deviation of the margins from the Gaussian can be studied separately and later.

The M 1965 model has a generic and specific aspect. Generic aspect: it introduces infinite memory into statistical modeling. Specific aspect: it introduces fractional Brownian motion (“FBM”), a process that has one significant parameter: the Hurst or Hölder exponent H satisfying $0 < H < 1$. The Wiener Brownian motion WBM is the atypical special case corresponding to the value $H = 1/2$. Early references on FBM are M & van Ness 1968 {H} and the papers by M & Wallis {H}; the use of FBM in

economics was pioneered in M 1970e, M 1971n, M 1971q, M 1972c and M 1973j. Samorodnitsky & Taqqu 1944 (Section 7.2) is one of many recent textbooks that discuss this process.

The original empirical test of long-run dependence, once again, used Pareto-style log-log plots, but did not apply them to the tail distribution but instead, to either the correlation or the spectrum. The latter take a very characteristic scaling form, described as “ $1/f$ ”, which is discussed in M 1997e (Chapter E6), M 1997h and M 1997n.

7.1. Concrete Justification for the Idea of Infinite Memory, through the Behavior of High-dimensional Systems, and a Metaphor from Physics

While FBM implies an infinite memory, it may be reassuring that a model based on FBM *need not* imply belief in action at a distance.

Indeed, consider a very high-dimensional system (physical or economic) that is Markovian when viewed in its full glory. Such a system’s one-dimensional or few-dimensional coordinates *need not* be Markovian at all. To think of them as following FBM involves no paradox whatsoever.

A useful metaphor is suggested at this point by the statistical physics of magnets. Infinite range dependence controlled by power-law expressions is the rule in systems in which actual interactions only occur between immediate neighbors, those systems must be of high enough dimensionality and be observed under conditions that physicists describe as “critical”.

Be that as it may, infinite memory in finance calls for explanation. It also proved to require a new frame of thinking, but, to my delight, was accepted more readily than infinite variance. To my knowledge, no writer went as far as P. H. Cootner did, when (as quoted in the Introduction) he described the M 1963 model as promising “blood, sweat and tears”.

7.2. Historical Digression: the Hydrology Connection

The intellectual path that led from LSM to FBM brings light on the similarities and differences between the M 1963 and M 1965 models, and therefore remains interesting. When I was a Visiting Professor of Economics at Harvard and it became known that I was able to deal with the “pathology” of price variation, I was flooded with examples of other pathologies. Most proved beyond my skills, but two exceptional “hits” led me far away from finance for a while.

In 1962, a pattern of very anomalous noise led to Berger & M 1963

{N5}, which implicitly introduces the notion of *fractal time* to which we shall turn in Section 8.

In 1963, hearing of the “Hurst puzzle” of hydrology (Hurst 1951, 1955, Hurst *et al.* 1965), I identified it immediately as a new example of scaling, and briefly believed that it required a straight replay of the M 1963 model. But this beautiful theory was soon demolished by a mere fact: while LSM was the proper tool to deal with price variability driven by long tails, yearly river discharges are *not* far from Gaussian. This led me to conclude that the Hurst puzzle was driven by the accumulation of variables that may even be Gaussian, yet exhibit serial correlation of infinite time span. The form of infinity raises many questions, but they are best discussed in Section 4 of Chapter E2 of M 1997e while tackling *ARMA* representations.

7.3. *The Widespread Confusion between the M 1963 and M 1965 Models*

Such confusion occurs despite the sharp differences between the Noah and Joseph Effects and the LSM and FBM processes. In the M 1963 (LSM) model, depending on which feature is singled out, fractal dimension is either $D_G = 2 - 1/\alpha$ or $D_T = \alpha$. In the M 1965 (FBM) model, depending on which feature is singled out, fractal dimension is either $D_G = 2 - H$ or $D_T = 1/H$. Even a competent mathematician sees between those two processes a number of parallelisms that some describe as “mysterious”.

Help is on the way. Section 4 of Chapter E6 of M 1997e generalizes the standard self-affine models further and presents the resulting family of possibilities in very graphic fashion. As a result, order and simplicity are restored, and confusion decreases.

The Noah and Joseph effects often coexist; this fact raises two issues. The following subsection sketches an effective way to disentangle the two effects’ contribution to a given record, and Section 8 sketches a versatile and effective way to build random processes that combine long-tails and long-dependence in “tunable” proportions.

7.4. *A Way to Disentangle the Noah and Joseph Contributions to a Record: R/S Analysis of Global Dependence, and its Application in Finance*

“R/S analysis” is concerned with the kind of global dependence that the eye perceives as clear-cut cycles having no determined periodicity. This statistical technique, very different from spectral analysis, originated in the work on river discharges that is described in M & Wallis 1968, 1969

a, b, c; see also M 1975h. This method started attracting wide attention, as exemplified by Feder 1988, and is one of the main topics of M 1997h. The details lie well beyond our scope, but it deserves a comment that paraphrases M 1970e.

"It is obviously important to know whether dependence in price change records vanishes, is positive or is negative. Unfortunately, the empirical investigations disagree, and all are unconvincing, because they invariably use statistical tools that imply that the underlying process is nearly Gaussian. Before any statistical test of dependence is used, its robustness with respect to infinite variance must be investigated. For this purpose, M & Wallis proposed *R/S* analysis, and I used it on financial data. It is not foolproof, in fact has not yet been extensively explored. But it should be added to the classical tools and promises to provide more applicable results. It measures the intensity of global *R/S* dependence by a single parameter *J*.

"From the viewpoint of *R/S*, all independent random processes and a variety of martingales behave identically in yielding $J = 1/2$. They can be called "*R/S* independent". A striking fact is that the notion of *R/S* independence is robust with respect to infinite variance. In the case of price changes, it can serve as a useful surrogate for market efficiency, which is far from easy to handle statistically.

"Using *R/S*, I analyzed interest rate series reported by Macaulay prices of commodities from various sources and series of daily and monthly returns on securities. I found that different kinds of "price" series fall into different categories.

"Certain prices, and also the rate of call money, exhibit global persistence with, for example, an exponent of $J = 0.7$. This result was expected: since call money was itself a tool of arbitraging, its price cannot itself be arbitrated to take advantage of inefficiency. Therefore, its behavior should follow closely that of the various exogenous quantities that affect the economy. There is strong evidence that economic time series other than price changes (Adelman 1965, Granger 1966) and various physical (e.g., climatic) triggers of the economy are globally persistent, and the *J* observed for call money rates is typical of exogenous economic quantities.

"At the other extreme, British Consols, cash wheat and some securities have *R/S* independent increments. The reason for this behavior is unclear. The data may be dominated by what may be called "market noise". However, spot commodity prices are not subject to thorough arbitraging. As a result, the absence of persistence in wheat is a puzzle. An explanation may be sought in institutional features; the arbitraging that is present in future prices may have an indirect effect on spot prices.

"Intermediate cases that exhibit a small degree of global dependence include prices of spot cotton and many securities. Closer investigations showed in many instances that the observed R/S dependence is wholly due to small price changes, which are both more difficult and less worthwhile to arbitrage. Large changes are practically R/S independent, even though they occur at highly non-independent (clustered) instants of time.

This and some of my other results leave many issues open. In particular, it is questionable whether or not the actually observed dependence is precisely compatible with efficiency. It is also unknown why there are so many differences between different series, and so many series in which the dependence is negligible".

A warning: No less than spectral analysis, R/S is a delicate statistical technique. There are rumors that the "Hurst's exponent" has become well-known in finance. However, recent developments reveal that R/S is an even more delicate technique than I believed in the 1960s. It is essential to use data that was *not* "prepared" in ways that unwittingly change them; for example, the elimination of short-term fluctuations judged *a priori* to be subject to their own rules may also eliminate the very non-periodic "long cycles" that R/S analysis is meant to reveal and investigate. Details are found in M 1997h, *Selecta* Volume H.

8. *Fractals in Finance, Stage III: the "M 1967" and "M 1972" Models; Trading Time and the "Noah-Joseph" Effect*

We are now ready to perform the crucial task of combining the non-Gaussian distribution of the M 1963 model with the dependence rule of the M 1965 model. The task took time and was not easy. Therefore, while this section is merely a preview of Chapter E6 of M 1997e, it is unavoidably more technical than the rest of this paper.

"Fractional Lévy flight" is a tempting obvious combination of scaling margins and long dependence. It is mathematically interesting, but fails to fit the actual records. Thirty years ago, its inadequacy set me to search for other broad methods in many different directions.

8.1. *Uniform or Variable Hurst-Hölder Exponents, the Distinction Between Physical (Clock) and Trading Time, and the Notion of Compound Process*

The Noah-Joseph combination to be described now is sufficiently general to include many important special cases: the B 1900 model, the M 1963 model without asymetry, and the M 1965 model. The key step is to introduce an auxiliary quantity called *trading time*. The term is self-explanatory and embodies two observations. While price changes over

fixed clock time intervals are long-tailed, price changes between successive transactions stay near-Gaussian over sometimes long time periods between discontinuities. Following variations in the trading volume, the time intervals between successive transactions vary greatly. This suggests that trading time is related to volume, but testing this empirical relation should be separated from an exploration of the model itself. Perhaps one could save Brownian motion by allowing price change to be due to extraneous impulses that are bunched in clock time.

To provide an alternative motivation of trading time, let us summarize very informally some properties of existing models concerning the “order of magnitude” of the price change Δx over a time increment Δt . Since Δt is assumed to be small, we deal with *local* behavior.

- For the B 1900 model, $\Delta x \sim \sqrt{\Delta t} = \Delta t^H$. The exponent is time invariant and $H = 1/2$.
- For the M 1963 and M 1965 models, $\Delta x \sim \Delta t^{1/\alpha}$ or $\Delta x \sim \Delta t^H$, respectively. The exponent $1/\alpha$ or H is again time invariant but $\neq 1/2$.

Unifractality versus multifractality. Because their scaling exponent is unique, the preceding models can be called *uniscaling* or *unifractal*. The generalization to which we now proceed can, by contrast, be called *multiscaling* or *multifractal*, because it consists in allowing the exponent to depend on t , and to be chosen among an infinity of possible distinct values.

Since we deal with local behavior of small Δt , large or small values of $H(t)$ express, respectively, that $x(t)$ varies slowly or rapidly near the instant t . Trading time is an alternative way of thinking about this variability of the exponent H . One imagines that $x(t)$ varies more or less uniformly in its own intrinsic time, but the latter varies non-uniformly in clock time.

The preceding two comments should suffice for motivation. As a strictly mathematical idea, every non-decreasing function $\theta(t)$ of physical time provides a formal representation of $Z(t)$ as a *compound* process $Z(t) = \tilde{Z}[\theta(t)]$. But the result will be a useless increase in complication, unless special circumstances prevail. I took both $\tilde{Z}(\theta)$ and $\theta(t)$ to be scaling, namely self-affine. To be practical, the only case I examined thus far is where $\tilde{Z}(\theta)$ and $\theta(t)$ are statistically independent (see Section 9). Specifically, I allowed $\tilde{Z}(\theta)$ to be a Wiener or fractional Brownian function, and $\theta(t)$ to be a fractal or multifractal time. As was hoped, the resulting generalization of Wiener Brownian motion provides a sensible approximation to interesting data that combine long tails and dependence. Let us take up the topic in historical sequence, which also corresponds to increasing difficulty.

8.2. Subordination and the "M 1967" Model: the "Symmetric M 1963" Model is Representable as a Wiener Brownian Motion in Fractal Time

The simplest form of compounding was pointed out formally by H. M. Taylor (Section 1 of M & Taylor 1967 {E21} and I went on to interpret it concretely (Section 2 of M & Taylor 1967 {E21}). The shortened form "M 1967 model" will be used, but it is not meant in any way to distract from the merits of Howard M. Taylor.

The M 1967 concerns the special case where $\theta(t)$ is a random function with *independent increments*; for historical reasons, it is called *subordinator*. The mathematical aspects of the notion of "subordination" (due to S. Bochner) are discussed in several places in Feller 1950 (Volume II). It came to play an important role in many aspects of fractal geometry, therefore the concrete aspects are discussed in detail in Chapter 32 of M 1982f {FGN}, where it is illustrated and interpreted in a variety of contexts.

Specifically, M & Taylor 1967 takes price to be a Wiener Brownian motion of trading time. In order for physical time to be a non-decreasing self-affine function of trading time, it is necessary for the graph of $t(\theta)$ to be the so-called Lévy devil staircase. This object is defined in Chapter 3 of M 1982f {FGN}, as the simplest randomized form of a Cantor devil staircase. The points where the staircase moves up form a "Lévy dust" characterized by an exponent that is the dust's fractal dimension, a concept to be sketched in Chapter E6. Each discontinuity of the inverted staircase corresponds to a step of the staircase and collapses a finite interval of trading time into an instant of physical time. Conversely, trading time followed as a function of physical time reduces to a series of mutually independent jumps of widely varying size. Price followed as a function of physical time also undergoes jumps.

Surprisingly, the above procedure simply reproduces the symmetric M 1963 model. The exponent α of the L-stable motion is "fed in" by choosing a Lévy staircase of dimension $\alpha/2$. (Some messy details will be discussed momentarily).

Devil staircases are standard examples of *self-affine fractals*, a concept described in Chapter E6 of M 1997e. Therefore, a trading time ruled by a devil staircase is called a *fractal time*. Section 4 of Chapter E2 of M 1997e mentions that Clark 1973 preserved subordination, but with a trading time that is not fractal. M 1973c argued against Clark's non-fractal substitute, but never implied that M & Taylor 1967 {E21} said the last word. Let us now proceed beyond.

A goal for generalizations of the M 1963 model: it is necessary to correct its unrealistic prediction, that large price changes are statistically independent ex-ante, therefore isolated ex-post. In the M 1967 model,

jumps with independent positions and amplitudes are inherent to the definition of subordination. Unfortunately, such jumps are unacceptable in the study of finance. Clear-cut bunching of large price changes is noted in M 1963b {E14}, but could not be seriously taken into account until a natural solution presented itself in the altogether different context of the study of turbulence.

A stepping stone towards generalization of the M 1963 model. Since it brings no new prediction or property, M & Taylor 1967 is best described as providing a representation". All too often, such formal representations are mathematically important, but of limited practical interest. A glowing exception, subordination opened the gates to generalizations to which we now proceed. They are new, even from the mathematical viewpoint.

An easily described generalization of the M 1967 model, replacing the Wiener Brownian motion by fractional Brownian motion. The result combines long tails and long-range dependence. It is defined by two main parameters: an α exponent that is twice the α exponent of the Lévy staircase, and the H exponent of the subordinated B_H . Little is known about it.

8.3. *Compounding and the M 1972 Model: Wieber Brownian Motion of Multifractal Time; the Turbulence Connection and the Paradoxical Character of the Perception of Infinite Memory by the Ear and the Eye*

The most direct replacement for subordination replaces fractal trading time by a construct that is more general and more richly structured (also, more complicated), called *multifractal time*.

The step from fractality to multifractality was first taken in my very first full publication on turbulence, M 1972j {N14}, to which we shall return momentarily. Today, every field takes this step near-automatically at some point in time, following a general pattern advocated in Chapter IX of M 1975o and in an entry on "relative intermittence" on p. 375 of M 1982f {FGN}. Broadly speaking, patterns that seem fractal in a first approximation tend on a second look to be multifractal.

Returning to M 1972j {N14}, it ends (p. 345 of the original) with the following words:

"The interplay ... between multiplicative perturbations and the lognormal and [scaling] distributions has incidental applications in other fields of science where very skew probability distributions are encountered, notably in economics. Having mentioned the fact, I shall leave its elaboration to a more appropriate occasion".

The concept first introduced in M 1972j {N14} is a family of many-parameter *multifractal functions* to be denoted by $M(t)$. They are non-

decreasing and continuous but non-differentiable. Their increments are called *multifractal measures*, and Figure 5 reproduces part of the original example in M 1972j {N14}. As is typical of the most interesting multifractal measures, the corresponding integral $M(t)$ is represented by a graph that is monotone increasing but lacks the flat steps characteristic of a devil staircase. It follows that the inverse of $M(t)$ has no jumps; like $M(t)$, it is continuous but non-differentiable.

Originally, the increments of $M(t)$ were meant to model the gustiness of the wind and other aspects of the intermittence of turbulence. An earlier *fractal* model of gustiness assumed that the wind comes in sharp isolated peaks. M 1972j {N14}, M 1974f {N15} and M 1974c {N16} put forward a more realistic *multifractal* picture of the wind's gustiness. After a delay of fifteen years, experiments (largely performed or supervised by K. Sreenivasan) confirmed the validity of that picture.

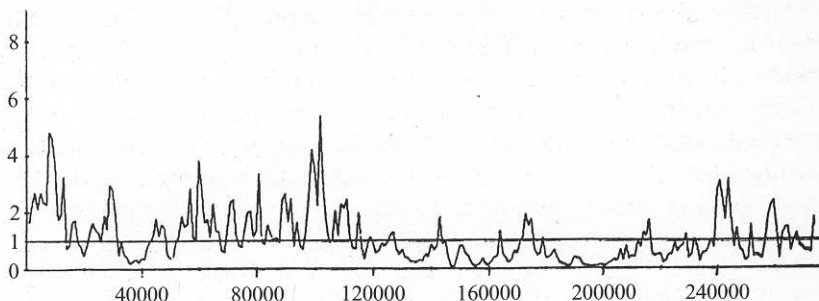


Figure 5

In addition, however, Figure 5 reminded me instantly of something entirely different, namely Figure 1 of M 1967j {E15}, which represents the variance of cotton price increments over successive time spans. After a long delay, this initial hunch proves to be an astonishingly good approximation. It was not elaborated until recently and is published for the first time in this book. The elaboration will, nevertheless, be called the “M 1972” model. Thus, my theoretical views of turbulence in the wind and the stock market were immediately and completely parallel.

Graphs analogous to Figure 1 of M 1967j {E15} are, of course, very familiar in finance, and their ubiquity motivates the “patchworks of quick fixes” called ARCH models, which are outlined and criticized in Section 4 of Chapter E2 of M 1997e. In the fractal context, on the contrary, the same resemblance immediately suggested a very different thought: a change in trading time from fractal to multifractal may generalize the M 1963 model.

Explanation, using acoustics, of the "unreasonable effectiveness" of infinite-memory models in accounting for bursts. ARCH-type models closely follow common sense. Even a casual look at diagram I of Figure 1 shows that large price changes are clustered. It is natural to attribute this fact to the presence of "high-frequency" serial dependence between price changes over neighboring time spans. Low-frequency serial dependence at a distance does not even come to mind.

On the contrary, my models start by accounting for very low frequencies, but they also succeed in accounting for the perceived high-frequency effects. Before we go on, we must establish that paradoxical claim is not absurd. This is best done by injecting yet another physical science connexion, namely, some phenomena called $1/f$ noises. Among them, the first to come to mind is the derivative of FBM, simply because it is Gaussian, but the most appropriate illustration a definitely non-Gaussian phenomenon called "flicker noise". As indicated by this name (or alternative names like "frying" and "popping"), the human ear perceives such a noise as a sequence of bursts separated by quieter periods. Hence, the ear seems to inform the brain that it deals with an "intermittent high-frequency phenomenon". Unfortunately, attempts to model this vague description failed. More importantly, the more objective spectral analysis yields a different diagnosis: a smooth spectrum concentrated in very low frequencies.

In a nutshell, the M 1972 model claims that the variance of $L(t, T)$ is a kind of flicker noise. As can be seen in M 1997n, I developed from 1964 to 1974 the technical know-how required to handle such noises, and the goal of the developments to be described from now on is to apply this know-how to finance.

8.4. *From Scaling to Multiscaling: from Marginal Distributions that "Collapse" to Long-tailed Distributions that Shorten under Averaging*

Let us return to a sober examination of the marginal distribution of price change. "Data collapse" is said to occur when $L(t, T)$ has the same distribution for all values of T , except for scale. This is predicted by the M 1963 model and observed in Figure 4. But other price series proved to behave in a different and more complicated fashion (Officer 1972). For them, the distribution of $L(t, T)$ is reasonably close to being L -stable for small T , but the tails become markedly shorter than predicted.

The psychological impact of those findings was surprisingly strong among students of speculation. The consensus became that, as T increases, $L(t, T)$ might eventually converge to the Gaussian, so that the M 1963 model does not matter much. As to the "transient" behavior

before the Gaussian is reached, it was to be handled by the ARCH-type models, and/or other "quick fixes".

No one could dispute that the observed drift is incompatible with the combination of scaling with statistical independence of price increments. Therefore, the presence of a drift means that it is necessary to face serial dependence. The point is that this can be done using a multifractal scenario developed from the above-quoted remarks in M 1972j {N14}.

This replacement led to the M 1972 model whose most striking prediction is as follows. As $T \rightarrow 0$, the distribution of $L(t, T)$ becomes increasingly sharp-peaked and long-tailed. Qualitatively, this prediction matches the empirical evidence.

A technical illustration of drift away from collapse. A key feature of multifractality concerns the scale factors $\sigma(q) = \{E[L^q(t, T)]\}^{1/q}$. In the fractal case, the scale factors for $q < \alpha$ are powers of T , with an exponent independent of q , which is why this case is called *uniscaling*. In the multifractal case, to the contrary, the exponents of the scale factors depend on q , which is why this case is called *multiscaling*.

A useful mental picture is suggested by the limit lognormal multifractals introduced in M 1972j {N14}. The picture consists in a sequence of lognormal variables Λ_σ such that $E\Lambda_\sigma = 1$, while $\log \Lambda_\sigma$ is characterized by an increasing variance σ^2 . The probability densities $p_\sigma(u)$ of those Λ_σ cannot be collapsed by using linear transformation. The M 1972 model predicts that the same is true of the distributions of $L(t, T)$ for different T . The resulting drift is slight if the exponent of the scale factor $s(q) = \{E[L^q(t, T)]\}^{1/q}$ increases little and slowly as with $q \rightarrow \infty$. But the drift's intensity can be "tuned" at will.

The careful reader may observe that, while the lognormal probability densities $p_\sigma(u)$ cannot be collapsed linearly, the corresponding expressions $\log p_\sigma(u)$ are easy to collapse. This observation is pretty much all there is to the multifractal function $f(\alpha)$ to be mentioned in Section 3.9 of Chapter E6 of M 1997e.

An alternative scenario behind the drift away from data collapse. The long term dependence characteristic of multifractals implements the following scenario, which I often heard mentioned, but do not recall seeing in print. In the case of statistical independence, a price increment of small probability p cannot be observed, unless the sample is at least equal to a few times $1/p$. But suppose that larger changes of $L(t, T)$ are strongly clustered for all values of T . If so, a price increment of probability p can only be observed on a sample containing any more than $1/p$ roughly independent values. A shorter sample should be expected to include far too few values in the tails.

When a sample for $t = 0$ to $t = t_{\max}$ is examined for increasing values of

T , the sample size t_{\max}/T decreases. Therefore, as T grows, the probability of hitting upon large deviations decreases. So does the histogram's tail.

8.5. *Simulation of the Multifractal M 1972 Model, and Empirical Tests*

As mentioned in the Preface, those tests were neither prompt, nor complete. Before they are described, it is good to restate a basic point already made in Section 2. The eye tells us that the behavior exemplified by Figure 1 is mimicked reasonably in Figure 2. We can now add an explanation: each line of Figure 2 is a separate implementation of a very simple "surrogate" to the M 1972 model – as explained fully in Section 4 of Chapter E6. "Fine tuning" the surrogate algorithm also tunes the output.

As to quantitative comparisons, I am overcommitted, and at present lack any competitive advantage in handling financial data. However, an exploratory study of foreign exchange rate changes is extremely promising. The results will be sketched in Chapter E6, after some technical tools are described. Full details will be published in free-standing form in M, Fisher & Calvet 1997, Calvet, Fisher & M 1997 and Fisher, Calvet & M 1997.

9. *Beyond all the Models Described in This Book*

Flaws of the preceding models should be immediately acknowledged.

Independently of whether fractional Brownian motion is followed in clock or multifractal time, its increments have a symmetrical distribution. Therefore, the M 1965, M 1967, and M 1972 models do not apply to cases where the distribution is clearly asymmetric, for example to cotton price changes.

The compound process in Section 8 assumes independence between trading time as function of clock time and price as function of trading time. It will be nice to go beyond independence.

For the above reasons and others, the M 1972 model does not claim to be the last word. In fact, my investigation of the combination of long tails and long dependence also tried out two very different approaches that remain little explored and may reward a fresh look.

M 1966b {E19} describes an interesting and surprisingly realistic form of rational market "bubbles". The key is a form of arbitrage whose input has short tails and long dependence, while its output is a martingale with long tails and remaining long dependence.

The "fractal sums of pulses" {FSP} processes are described in M 1995n and several papers I co-authored with R. Cioczek-Georges, listed in the Biography of M 1997e. The FSP are used in Lovejoy & M 1985 {H} to simulate the shapes of clouds, and also show promise in finance.

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