Convergence of a random walk to Brownian motion scaling

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Abstract

Random walks converge on the scaling of Brownian motion, but the transition between the two is not well understood. Using the distribution of gap sizes in a random walk trail, we show numerically using very large walks that the asymptotic scaling is approached very slowly. The random walk scaling (leading to $D = 2$) is reached only at large scales.

Brownian motion trails have a dimension $D = 2$. The random walk has a finite step length, but approach the properties of Brownian motion at large sizes. However, the approach to the Brownian limit is not well understood. The length of non-grid-bound random walks as a function of resolution (step length) has been measured[3] to be about 1.65; however, the fit to the data is over only about 1 order of magnitude. A similar numerical experiment with more data [4] found the result could not be distinguished from D=2, though the fit was still only over a little more than 1 order of magnitude.

Here we measure the properties of much larger walks in an embedding dimension $D_E = 2$ using the distribution of gap sizes. The dimension D of fractal "nets", such as Brownian motion trails, can be determined using the distribution of the areas of their gaps since these areas a are distributed as

$Pr(A > a) \propto a^{-D/2}$[1]. Numerical simulations of very large random walks (8×10^8 steps) on lattices show (fig. 1) that the Brownian limit giving $D = 2$ is reached only at very large sizes. The gaps are distributed as a^{-x} where $x < D/2$ throughout the range measured (gap areas up to 10^6), with D measured to be about 1.85 over 3 orders of magnitude. At large areas, the slope is consistent with $D = 2$ for about one order of magnitude, though this increased slope may also be due to the cutoff of large areas by the lattice size.

Gap areas were measured with three different counts: number of "tiles", that is, lattice squares enclosed by the gap (a_T); number of bonds in the interior of the gap (a_I); and number of bonds connected to sites in the interior of the gap, that is, sites not touching the gap's boundary (a_S). a_I gives a power law over the widest range; a_T and a_S deviate from power-law scaling somewhat below about $a = 1000$; above this all three are similar.

References

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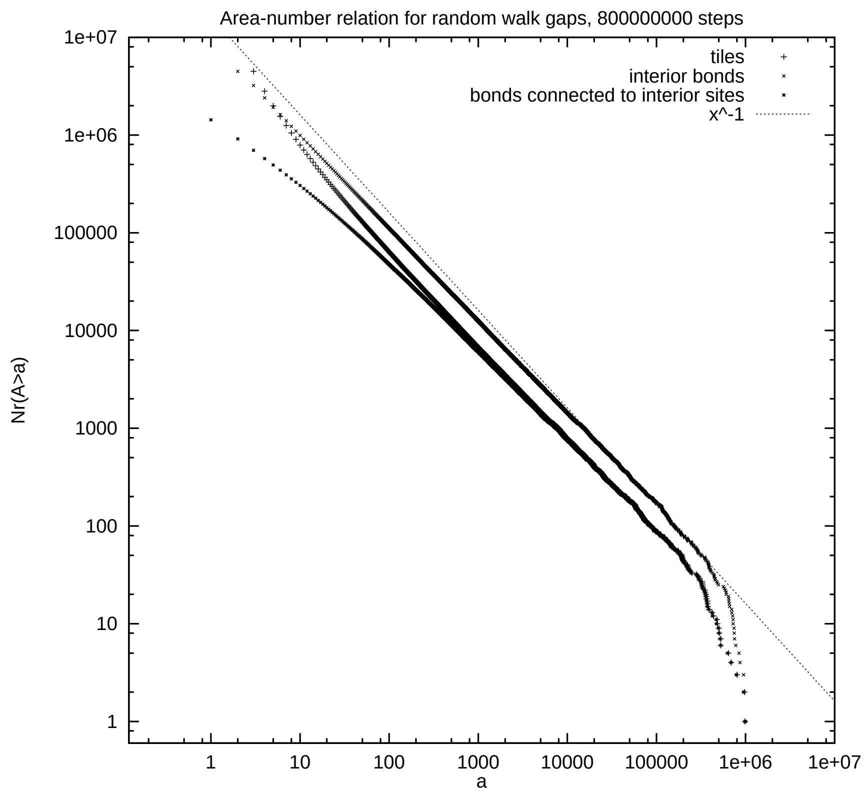
(a)


Area-number relation for random walk gaps, 800000000 steps
tiles
interior bonds
bonds connected to interior sites
x^-1
Nr(A>a)
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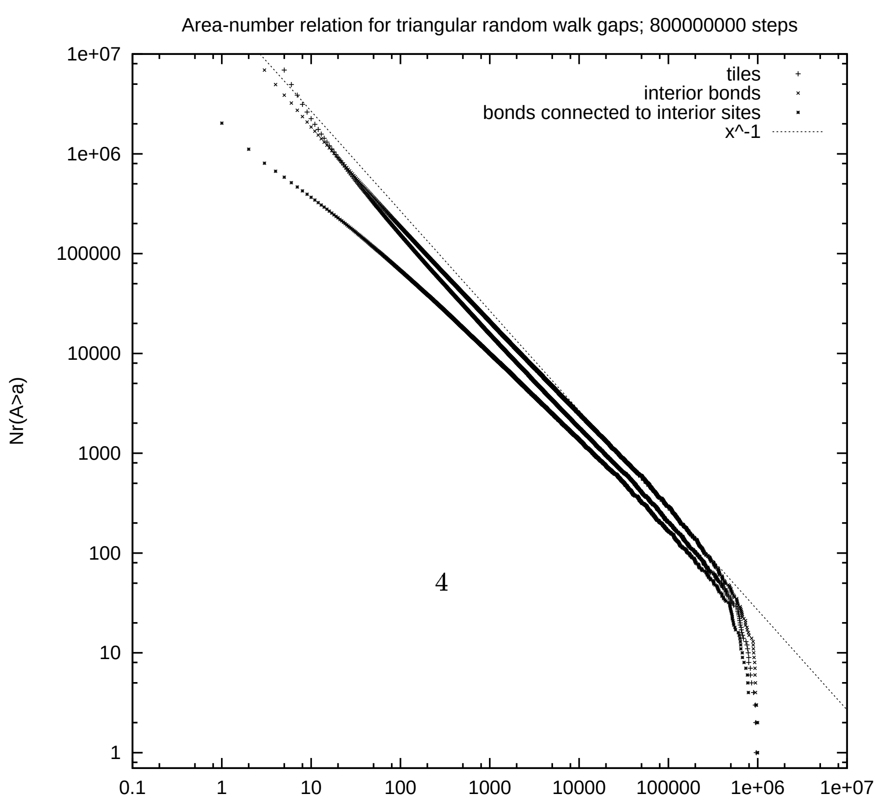
(b)


Area-number relation for triangular random walk gaps; 800000000 steps
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interior bonds
bonds connected to interior sites
x^-1
Nr(A>a)
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