Some Mathematical Questions Arising in Fractal Geometry

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Abstract. The bulk of this text sketches diverse questions of pure mathematics that fractal geometry raised over the years. Some are new; others are broadly-based or fully-fledged conjectures that resist repeated efforts to answer them. A few can be understood by a good secondary-school student, while others are delicate or technical. Their perceived importance ranges from high to low, but they are alike in three ways. First, they did not arise from earlier mathematics, but in the course of practical investigations into diverse natural sciences, some of them old and well-established, others newly revived, and a few of altogether new. Second, they originate in careful inspections of actual pictures generated by computer. Third, they involve in essential fashion the century-old mathematical "monster shapes" that were for a long time guaranteed to lack any contact with the real world. A last section describes a case where fractal geometry helped with a mathematical question that had been around for a century.

1 Introduction

For the reasons listed in the abstract, the questions raised in this paper bear on an issue of great consequence. Does pure (or purified) mathematics exist as an autonomous discipline, one that can – and ideally should – adhere to a Platonic ideal and develop in total isolation from "sensations" and the "material" world? Or, to the contrary, is the existence of totally pure mathematics a myth?

The role of "sensations." My work is dominated by the role of fully-fledged pictures that are as detailed as possible and go well beyond mere sketches and diagrams. Their original goal was modest: to gain acceptance for ideas and theories that were developed without pictures but were slow to be accepted because of cultural gaps between fields of science and mathematics. But those pictures then went on to help me and many others generate new ideas and theories. Many of these shapes strike everyone as being of exceptional and totally unexpected beauty. Some have the beauty of the mountains and clouds they are meant to represent; others are abstract and seem wild and unexpected at first, but after brief inspection appear totally familiar. In front of our eyes, the visual geometric intuition built on the practice of Euclid and of calculus is being retrained with the help of new technology.

Pondering these pictures proves central to a different philosophical issue. Does the beauty of these mathematical pictures relate to the beauty that a mathematician rooted in the twentieth century mainstream sees in his trade after long and strenuous practice? My lectures often underline these questions, by showing in full colors what certain mathematical shapes really look like. By now, these pictures have become so ubiquitous that one will suffice here.

The relation between pure mathematics and the "material" world. Everyone agrees that an awareness of physics, numerical experimentation and geometric intuition are very beneficial in some branches of mathematics, but elsewhere physics is reputed to be irrelevant, computation powerless, and intuition misleading. The irony is that history consistently proves that, as branches or branchlets of mathematics develop, they suddenly either lose or acquire deep but unforeseen connections with the sciences – old and new. As to numerical experimentation – which Gauss had found invaluable, but whose practice was waning until yesterday – it has seen its power multiply by a thousandfold thanks to computers, and later, to computer graphics.

In no case that I know is this irony nearly as intense as in fractal geometry, a branch of learning that I conceived, developed and described in my book *FGN*. I put it to use in models and theories relative to diverse sciences, and it has become widely practiced. A "Polish school" of mathematics had viewed itself as devoted exclusively to *Fundamenta*, added mightily to the list of monster shapes, and greatly helped create a gulf between mathematics and physics. Specifically ironical, therefore, is the fact that my work, that of my colleagues, and now that of many scholars, made those monster shapes, and new shapes that are even more "pathological," into everyday tools of science.

This article uses freely the term *fractal*; I coined it in 1975 from the Latin word for "rough and broken up," namely *fractus*, and it became generally accepted. Loosely, a "fractal set" is one whose detailed structure is a reduced-scale (and perhaps deformed) image of the overall shape, hence the term "self-similar." When the reduced scale images are distorted by being reduced different amounts in different directions, the fractal is "self-affine."

2 Complex Brownian bridge; Brownian cluster and its boundary; the self-avoiding plane Brownian motion

We begin with the open conjecture that is easiest to state and to understand.

Background. The Wiener Brownian motion $B(t)$ is a random process whose increments $B(t+h) - B(t)$ are Gaussian random variables with mean 0 and variance h, and are independent over disjoint intervals. It is well known that $B(t)$ is statistically self-affine in the sense that

$$Pr\{B(t+h) - B(t) \leq b\} = Pr\{B(s(t+h)) - B(st) \leq \sqrt{s}b\},$$

and the same is true of joint probability distributions for all finite collections of time intervals h_j.

Assuming $B(0) = 0$, a *Brownian bridge* $B_{bridge}(t)$ is a periodic function of t, of period 2π, given for $0 \leq t \leq 2\pi$ by

$$B_{bridge}(t) = B(t) - (t/2\pi)B(2\pi).$$

In distribution, $B_{bridge}(t)$ is identical to a sample of $B(t)$ conditioned to return to $B(0) = 0$ for $t = 2\pi$. It is the sum of Wiener's trigonometric series, whose n^{th} coefficient is G_n/n, where the G_n are independent reduced Gaussian random variables.

Definitions. Take $B_{bridge}(t)$ to be complex of the form $B_r(t) + iB_i(t)$ and define a *Brownian plane cluster Q* as the set of values of $B_{bridge}(t)$. This non-traditional concept is the map of the time axis by the complex form of $B_{bridge}(t)$. The classical map of the time axis by $B(t)$ is everywhere dense in the plane, and the map of a time interval by $B(t)$ is an inhomogeneous set. In contrast, when the origin Ω of the frame of reference belongs to Q, all the probability distributions concerning Q are independent of Ω; therefore Q is a *conditionally homogeneous* set.

The *self-avoiding planar Brownian motion* $\tilde{Q}$ is defined in *FGN* as being the closed set of points in Q accessible from infinity by a path that does not intersect Q.

Unanswered Conjecture. The set $\tilde{Q}$ has a fractal dimension of 4/3, in some suitable sense: Hausdorff-Besicovitch, or perhaps Bouligand ("Minkowski"), Tricot ("packing"), and/or other.

Comment. The original illustration of Q in Plate 243 of *FGN* is reproduced as Figure 1. It looked to me like an island with an especially wiggly coastline, hence visual intuition nourished by experience in the sciences suggested $D \approx 4/3$. This value was confirmed by our direct numerical tests, and by further more recent indirect numerical tests by W. Werner.

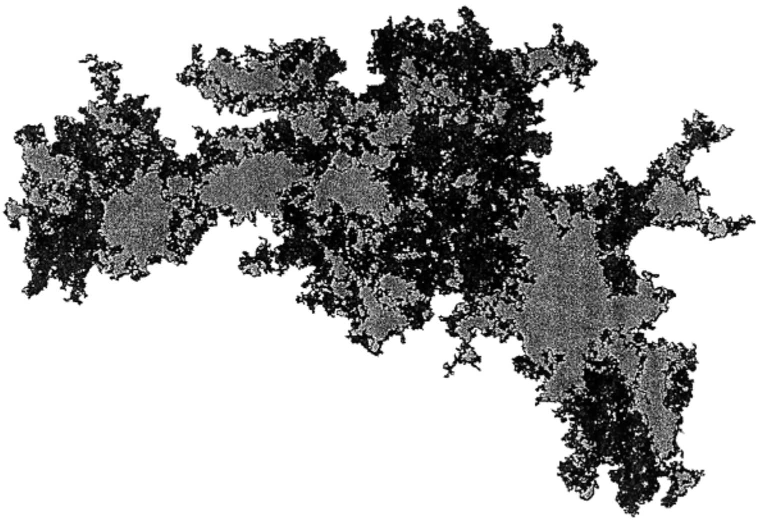

Figure 1

Literature: It is extensive: Peres, Jones, and includes Pimenkel & Peres, *J. Funct. Analysis*: 143, 1997, 309; Lawler & W. Werner, *Electronic Comm. Proba*: 1, 1996, 19; also a 1997 Orsay Report.

Comments on the dimension 4/3 and self-avoidance. There are two reasons for the term "self-avoiding Brownian motion:" a) $\tilde{Q}$ does not self-intersect and b) its conjectured dimension 4/3 is found in the self-avoiding random walk (SARW) on a lattice. This value 4/3 is unquestioned but was obtained by analytic arguments that are geometrically opaque; its interpretation as a dimension implies yet another unproven conjecture.

Squigs and a wide open issue that combines fractals and topology. SARW is difficult to study because it is obtained by extrapolation. *FGN* (Chapter 24) introduced a class of recursive constructions, *squigs*, that create self-avoidance by interpolation. The simplest is of dimension log 2.5 / log 2 $\approx$ 1.3219...; my heuristic argument was confirmed by J. Peyrière. I suspect the discrepancy between 4/3 and 1.3219... follows from the fact that squigs involve a discrete and recursive subdivision of the plane into triangles. Viewing this discrepancy as of secondary importance, I suspect that self-avoidance is linked in a profound and intrinsic way to the dimension 4/3. The nature of this link is a mystery and a challenge.

3 Explosive multiplication of fractal constructions

The number of distinct fractal constructions was small before fractal geometry became organized but grew very rapidly as fractals became tools in the sciences and favorites in computer graphics. Fractional Brownian motion (*SH*) and multifractal measures (*SN*) led to a rich mathematical literature, but not the other new constructions.

From stable processes and fractional Brownian motion to fractal sums of pulses. Brownian motion was generalized in two deeply different ways by Lévy's stable processes (LSM) and fractional Brownian motions (FBM). The LSM depended on a parameter α, with $0 < \alpha \leq 2$ and $\alpha = 2$ yielding the Brownian as a limiting case. They are investigated, among many other places, in *SN*. The FBM depends on a parameter H, with $0 < H < 1$ and $H = 1/2$ yielding the Brownian as a critical case. By the definition of $B_H(t)$, the increment $B_H(t) - B_H(t')$ is a Gaussian random variable of expectation 0 and standard deviation $|t - t'|^H$. The FBM are investigated, among other places in *SH*.

Given the deep differences, the formal analysis observed between the studies of LSM and FBM are often perceived as surprising. They became very natural when both families are imbedded in a far broader family, the "fractal sums of pulses" (FSP). The FSP also allow a variety of additional behaviors that are useful in science and may be of mathematical interest. The best reference will be *SH*.

Multifractal measures. The pioneer papers on multifractal measures are reprinted in *SN*. The topic is too rich to be dwelt upon here, but it is useful to note that a multifractal measure is, above all, described by a function $f(\alpha)$ of the parameter α. (The derivation of a function equivalent to $f(\alpha)$, given in my original 1974 paper, applied to multifractals obtained by an infinite product of multiplications and relied on the Cramer theory of large deviations.)

Fractional Brownian motion in multifractal time, and its use in financial modeling. Chapter 6 of *SE*, not previously published, offers a new model of variation of prices. Price is taken to be a fractional Brownian motion B_H, as followed in a "trading time," θ, which is a multifractal function $\theta(t)$ of clock time, namely, the integral of a random multifractal measure. That is, $P(t) = B_H[\theta(t)]$. At this stage, B_H and θ are assumed to be statistically independent. This process is specified by H (a Hölder exponent) and the properties of $\theta(t)$ beginning with its $f(\alpha)$ spectrum. This process was found to fit diverse financial data very well and a rigorous foundation was provided by R. Reidi (to be published). From most other viewpoints, it is wide open for exploration.

The notion of "states of randomness," from mild to wild. Chapter 5 of *SE* argues that, while probability theory has unified foundations, it is best to consider a function's variability as belonging to one of several distinct "states of randomness." The law of large numbers combined with the central limit theorem characterizes "mild" randomness. The processes mentioned in preceding sections, namely LSM, FBM, FSP and multifractal measures, as functions characterize "wild" randomness. So far, this distinction has not been adequately discussed from the mathematical viewpoint.

4 The many forms of fractal dimension

The original Hausdorff-Besicovitch dimension D_{BH} is impossible to measure because it contains the operation "inf." It also involves a limit, but this is not a problem in the case of self-similar or self-affine shapes.

Self-similar sets. In that case, the many definitions of fractal dimension yield identical values. A set S is self-similar if it is constructed recursively and its generator consists of N copies of itself, the i^{th} copy S_i being obtained from S by a similarity with contraction factor r_i. The dimension calculation is relatively simple. Under a mild condition (the "open set" condition), the fractal dimension is the solution δ of the Moran generating equation

$$\sum_{i=1}^{N} r_i^{\delta} = 1.$$

Self-similar multifractal measures and negative dimensions. As mentioned, multifractal measures are largely specified by a function $f(\alpha)$, hence by an infinite number of parameters. When $f > 0$, f is a fractal dimension, for example in the sense of Hausdorff–Besicovitch. When $f < 0$, f takes an altogether different new role, as a measure of "degree of emptiness." (Mandelbrot, *J. Fourier Analysis and Applications* (Kahane issue), 1995, 409). Negative dimensions amply deserve closer study.

Self-Affine Sets. When the transformation of S into S_i is an affinity, the evaluation of D_{HB} was successful in a surprisingly small number of cases. Contributors include McMullen, Bedford, Falconer, Peres, Kenyon, Lalley, and Gatzouraus.

Furthermore, the many alternative definitions of fractal dimension yield values that differ from D_{HB} and from one another. In particular, the concepts of local and global dimension, which coincide in the self-similar case, greatly differ in the case of self-affinity.

Richard Kenyon, 1998

See Mandelbrot in *Fractals in Physics* (E. Pietronero & E. Tosatti, eds., 1986, reprinted in *SH*). The global notions of dimension pose many mathematical issues.

All these computations suggest that, while the notion of fractal dimension can be defined under wide conditions, its "natural domain" of practical relevance centers around self similarity.

5 The many forms of the Hölder exponent

In the case of the graph of a self-affine function, the most "natural" quantitative description is not provided by a dimension, but by diverse forms of an exponent first introduced by Hölder and Lipschitz and later by the hydrologist H. E. Hurst. The variable α in the multifractal function $f(\alpha)$ is a Hölder exponent. Chapter 6 of *SE* and chapter 1 of *SN* show that the original definitions have, in response to concrete needs, branched in diverse directions.

Particularly great variety is found in the fractional Brownian motions of multifractal time; as already mentioned, *SE* put these as models of price variation. They are of unbounded variation. More generally, define their q^{th} variation by using the same formula as the ordinary variation, except that $|dP|$ is replaced by $|dP|^q$. Then the q^{th} variation is infinite for $q < 1/H$ and vanishes for $q > 1/H$. The value $q = 1/H$ is "critical" and defines the tau dimension D_τ, a notion that generalizes to all processes. The inverse $1/D_\tau$ is yet another form of Hölder's exponent. Its properties deserve careful mathematical study.

6 Other tools of fractal analysis: new or old but obscure

Careful analysis brings in many fractal tools beyond dimension.

Sierpinski curves and Urysohn-Menger ramification. Two ancient decorative designs occur in Sierpinski's investigations in the 1900's: one became known as the "carpet," and the second I called the "gasket." Sierpinski used the carpet to show that a plane curve can be "topologically universal," that is, contain a homeomorphic transform of every other plane curve. The construction starts with a square, divides it into nine equal subsquares and erases the middle one, which I call a "trema" ($\tau\rho\eta\mu\alpha$ is the Greek term for "hole"). One proceeds in the same fashion with each remaining subsquare, and so on ad infinitum. As to the "gasket," Sierpinski used it to show a curve can have branching points everywhere. The construction starts with an equilateral triangle, divides it into four equal subtriangles and erases the middle one as trema. One proceeds in the same fashion with each remaining subtriangle, and so on ad infinitum.

During the 1920's, the distinction between the carpet and the gasket became essential to the theory of curves. Piotr Urysohn and Karl Menger took them as prime examples of curves having, respectively, an infinite and a finite "order of ramification."

FGN quotes influential mathematicians for whom the "gasket" gave prime evidence that geometric intuition is powerless, because it can only conceive of branch points as being isolated, not everywhere dense. Contemplation of the Eiffel Tower cast doubts about this contention, in fact, Gustave Eiffel himself wrote (as I interpret him) that he would have made his tower even lighter, with no loss of strength, had the availability and cost of finer materials allowed him to increase the density of double points. From the Eiffel Tower to the Sierpinski gasket is an intellectual step that one's intuition can be trained to take.

The theory of curves that studies carpets, gaskets and the order of ramification became a stagnant corner of mathematics. Where can one find the latest facts about these notions? The surprising answer is that, after I introduced them in the statistical physics of condensed matter, physicists came to view these notions as "unavoidable." Once ridden of the cobwebs of abstraction, they prove to be very practical and enlightening geometric tools to work with. Physicists make them the object of scores of articles, and invent scores of generalizations that mathematicians did not need in 1915.

A new fractal tool: lacunarity. As is well known, the most standard construction of a Cantor dust proceeds recursively as follows. The "initiator" is the interval [0, 1]. Its first stage ends with a generator made of N subintervals, each of length r. In the second stage, each generator interval is replaced by N intervals of length r^2, etc.... The resulting limit set arose in the study of trigonometric series, but first attracted wider interest because of its topological and measure-theoretical properties. From those viewpoints, all Cantor dusts are equivalent. Much later, Hausdorff introduced his generalized dimension; this and every other definition of dimension yield $D = \log(N)/\log(1/r)$. The value of dimension splits the topological Cantor dusts into finer classes of equivalence parameterized by D.

Fractal geometry showed those classes of equivalence to be of great concrete significance. In due time, the needs of science, rather than mathematics, required an even finer subdivision. To pose a problem, consider the Cantor-like constructions stacked in

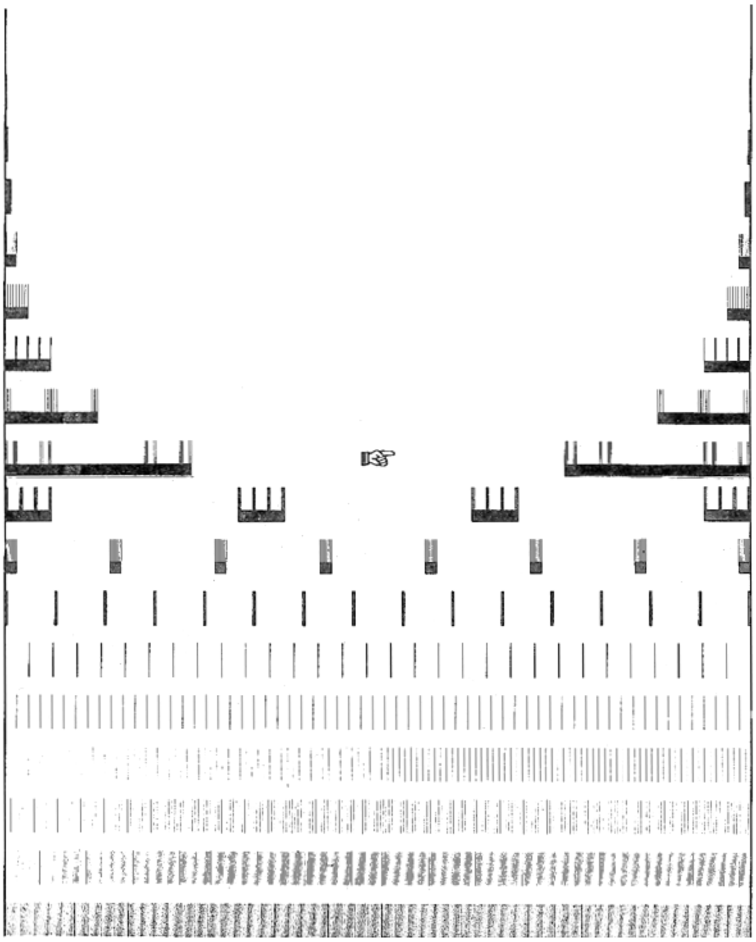

Figure 2

Figure 2. In the middle line, $N = 2$ and $r = 4^{-1}$; k steps below the middle line, $N = 2^k$, $r = 4^{-k}$ and the generator intervals are uniformly spaced; k steps above the middle line, $N = 2^k$, $r = 4^{-k}$, again, but the generator intervals are crowded close to the endpoints of [0, 1]. The Cantor dusts in this stack share the common values $D = 1/2$, but look totally different. The Latin word for hole being *lacunar*, motion down the stack (or up) is said to correspond to decreasing (increasing) *lacunarity*.

Challenge. As $k \to \infty$, the bottom line becomes "increasingly dense" in [0, 1], and the top line "increasingly close to two dots." Provide a mathematical characterization of this "singular" passage to the limit.

Second challenge. FGN, Chapters 33 to 35, describes and illustrates several constructions that allow a control of lacunarity. However, for the needs of both mathematics and science, the differences between the resulting constructs must be quantified. The existing studies of this quantification show that it is not easy and also not unique. Special complications occur when all the reduction ratios are identical, like in Figure 2. Of the alternative methods investigated in the literature, one is based on the prefactor relation $M(R) = FR^D$ that yields the mass $M(R)$ contained in a ball of radius R. Another method is based on the prefactor in the Minkowski content.

A third method has the advantage that defines a neutral level of lacunarity that separates positive and negative levels.

On the line, this level is achieved by any randomized Cantor dust S with the following property. Granted that any choice of origin Ω in S divides the line into a right and a left half line, lacunarity is said to be neutral when the intersections of S by those half lines are statistically independent. Increasingly positive (resp. negative) correlations are used to express and measure increasingly low (resp. high) levels of lacunarity. These notions will be used in the sections that follow.

7 Major fractal clusters in statistical physics

While Brownian motion is fundamental in physics as well as in mathematics, the Brownian clusters of the first section are a mathematical curiosity. However, the property of fractality is shared by all the major real clusters (turbulence, galaxies, percolation, Ising, Potts) and all the major real interfaces (turbulent jets and wakes; metal and glass fractures; diffusion fronts). Each of these categories raises numerous open mathematical questions, of which a few will be tested.

Percolation clusters at criticality. (D. Stauffer & A. Aharony. *Introduction to Percolation Theory.* Second edition. London: Taylor & Francis.) Take an extremely large lattice of tiles. Each tile is chosen at random: with the probability p, it is made of vinyl and with the probability $1 - p$, of copper. Allow electric current to flow between two tiles if they have a side in common. A “cluster” is defined as a collection of copper tiles such that electricity can flow between two arbitrary points in the cluster. For an alternative, but equivalent, construction, at the center of every tile define a random “relief function” $R(p)$ whose values are independent random variables uniformly distributed from 0 to 1. If this relief is flooded up to level p, each cluster stands out as a connected “island.” Physicists conjectured, and mathematicians eventually proved, that there exists a “critical probability” p_C, such that a connected infinite island, or connected infinite conducting cluster, almost surely exists for $p < p_C$, but not for $p > p_C$.

The geometric complication of percolation clusters at criticality is extreme, and many of the basic conjectures do not arise from pure thought, but from careful examination of graphics.

Open conjecture A. Take an increasingly large lattice and resize it to be a square of unit side. At p_C, the infinite cluster converges weakly to a “limit cluster” that is a fractal curve.

Open conjecture B. The fractal dimension of this limit cluster is 91/48. This is the value obtained from a partly heuristic “field theoretical” argument that yields characteristic exponents.

Open conjecture C. The limit cluster is a finitely ramified curve in the sense of Urysohn-Menger.

Open conjecture D. Linear cross-sections of the limit cluster are Lévy dusts, as defined in *FGN*. Experimental evidence is found in Mandelbrot & Stauffer, *J. Physics*: A 28, 1995, L 213 and Hovi et. al. *Phys. Rev. Lett.*: 77, 1996, 877.

The Ising model of magnets at the critical temperature. At each node of a regular lattice, the Ising model places a spin that can face up or down. The spins interact via forces between neighbors. By themselves, these forces create an equilibrium (minimum potential) situation in which all the spins are either up or down. In addition, the system is in contact with a heat reservoir, and heat tends to invert the spins. When the temperature T exceeds a critical value T_C, heat overwhelms the interaction between neighbors. For $T < T_C$, local interactions between neighbors create global structures of greatest interest. My work touched upon several issues in the shape of the up (or down) clusters at criticality.

Long open implicit question: Beginning with Onsager, it is known that in Euclidean space R^E the necessary and sufficient condition for magnets to exist is that $E > 1$. There are innumerable mathematical differences between the R^E for $E = 1$ and $E > 1$. Identify differences that matter for the existence of magnets.

Partial answer: The specific examples of the Sierpinski curves and of related fractal lattices suggest that magnets can exist if and only if the order of ramification is infinite. (*FGN*, p. 139; Gefen et al, *Phys. Rev. Lett.*: 45, 1980, 855).

Conjecture: The above answer is of general validity.

Unanswered challenge. Rephrase the criterion of existence of magnets from the present and highly computational form, to a direct form that would give a chance of proving or disproving the preceding conjecture.

Actual geometric implementation of the fractional-dimensional spaces of physics. Physicists are very successful with a procedure that is mathematically very dubious. They deal with spaces whose properties are obtained from those of Euclidean spaces by interpolation to "noninteger Euclidean dimensions." The dimension may be $4-\epsilon$ or $1+\epsilon$, where ϵ is in principle infinitesimal but is occasionally set to $\epsilon = 1$. Calculations are carried out, in particular, expansions are performed in ϵ, and at the final stage, the "infinitesimal" ϵ is set to be the integer. Mathematically, these spaces remain unspecified, yet the procedure turns out to be extremely useful.

Mathematical challenge: Show that the properties postulated for those spaces are mutually compatible, show that they do (or do not) have a unique implementation; describe their implementation constructively.

Very partial solution: A very special example of such space has been implemented indirectly (*FGN*, second printing, p. 462; Gefen et al, *Phys. Rev. Lett.* 50, 1983, 145). We showed that the postulated properties of certain physical problems in this space are identical to the *limits* of the properties of corresponding problems in a Sierpinski carpet whose "lacunarity" is made to converge to 0, in the sense that it tends to 0 as one moves down the stack on Figure 2.

8 The origin of fractality in partial differential equations

To establish that many features of nature (and, as shown in *SE*, also of the Stock Market!) are fractal was a daunting task to which a large portion of *FGN* is devoted. New and often important examples keep being discovered, but the hardest present challenge is to discover the *causes* of fractality. Some cases remain obscure, but others are reasonably clear.

Thus, in the case of the physical clusters discussed in the preceding section, fractality is the geometric counterpart of scaling and renormalization, that is, of the fact that the analytic properties of those objects follow a wealth of "power-law relations." Many mathematical issues, some of them already mentioned, remain open, but the overall renormalization framework is firmly rooted. Renormalization and resulting fractality also occur in arguments that involve the attractors and repellers of dynamical systems. Best understood is renormalization for quadratic maps. Feigenbaum and others considered the real case. For the complex case, renormalization establishes that the Mandelbrot set contains infinitely many small copies of itself.

Unfortunately, additional examples of fractality proved to be beyond the scope of the usual renormalization. A notorious case concerns the diffusion-limited aggregates (DLA). Yet, another source that covers many very important occurrences of fractality led me to a very broad challenge-conjecture which was stated in *FGN*, Chapter 11, and which we now proceed to discuss. Are smoothness and fractality doomed to coexist? *A quandary.* It is universally granted that physics is ruled by diverse partial differential equations, such as those of Laplace, Poisson, and Navier-Stokes. All differential equations imply a great degree of local smoothness, even though closer examination shows isolated singularities or "catastrophes." To the contrary, fractality implies everywhere dense roughness and/or fragmentation. This is one of the several reasons why fractal models in diverse fields were initially perceived as being "anomalies" that stand in direct contradiction with one of the firmest foundations of science.

A conjecture. There is no contradiction at all; in fact, fractals arise unavoidably in the long time behavior of the solution of very familiar and "innocuous"-looking equations. In particular, many concrete situations where fractals are observed involve equations having free and moving boundaries, and/or interfaces, and/or singularities. As a suggestive "principle," *FGN* (Chapter 11) described the possibility that, under broad conditions that largely remain to be specified, these free boundaries, interfaces and singularities converge to suitable fractals. Many equations were examined from this viewpoint, but this paper will limit itself to two examples of critical importance.

The large scale distribution of galaxies: Newton's law as generator of fractality. *Background.* The near universally held view is that the distribution of galaxies is homogeneous, except for local deviations.

However, (*FGN*, Chapter 9), philosophers or science fiction writers played with the notion that the distribution is hierarchical, in a way unknowingly patterned along a spatial Cantor set. For a variety of reasons, hierarchical models were dismissed as unrealistic and largely forgotten: They are excessively regular and necessarily imply that the Universe has a center. Last but not least, they predict nothing, that is, have no property that was not put in beforehand, and raise no new question.

Conjecture that the distribution of galaxies is properly fractal. (*FGN*, Chaps. 9 and 33 to 35.) This conjecture results from a search for invariants that was central to every aspect of my construction of fractal geometry. Granted that the distribution of galaxies certainly deviates in some ways from homogeneity, two broad approaches were tried. One consists in correcting for local inhomogeneity by using local "patches." The next simplest global assumption is that the distribution is non-homogeneous but scale-invariant. I chose to follow up this assumption, while excluding the strict hierarchies. A surprising and noteworthy finding rewarded a detailed mathematical and visual investigation of sample sites generated by two concrete constructions of random fractal sets. Being random, their self-similarity can only be statistical, which may be viewed as a drawback. But a more than counter-acting strong asset is that the self-similarity ratio can be chosen freely. It is not restricted to powers of a prescribed r_0, that is, the hierarchical structure is not a deliberate and largely arbitrary input. Quite to the contrary, the existence of clear-cut clusters are an unanticipated property of the construction. The details are given in *FGN*. The first construction is The Seeded Universe, based on a Lévy flight. Its Hausdorff-dimensional properties were well known. Its correlation properties (Mandelbrot, *C. R. Acad. Sci. Paris*: 280 A, 1975, 1075) are nearly identical to those of actual galaxy maps. The second construction is The Parted Universe, which is obtained by subtracting from space a random collection of overlapping sets, tremas. Either construction yields sets that are highly irregular and involve no special center, yet exhibit a clear-cut clustering that was not deliberately inputted. They also exhibit "filaments" and "walls," which could not possibly have been inputted, because I did not know that they had been observed.

Conjecture that the observed "clusters," "filaments" and "walls," need not be explained separately, but necessarily follow from "scale free" fractality. This subtitle consists in conjecturing that the properties that it lists do not result from unidentified specific features of the models that have actually been studied, but follow as consequences from a variety of unconstrained forms of random fractality.

In the preceding title and the sentence that elaborates it, the word "conjecture" cannot be given its strict mathematical meaning, until a mathematical meaning is advanced for the remaining terms.

Lacunarity. A problem arose when careful examination of the simulations revealed a clearly incorrect prediction. The simulations revealed in the Seeded Universe proved to be visually far more "lacunar" than the real world. This notion, which was already mentioned, means that the simulations show the holes larger than in reality. The Parted Universe model fared better, since its lacunarity can be adjusted at will and fitted to the actual distribution.

A lowered lacunarity is expressed by a positive correlation between masses in antipodal directions. Testing this specific conjecture is a challenge for those who analyze the data.

Conjectured mathematical explanation of why one should expect the distribution of galaxies to be fractal. Consider a large array of point masses in a cubic box in which opposite sides are identified to form a three-dimensional torus. The evolution of this array is a problem that obeys the Laplace equation, with the novelty that the singularities of the solution are the positions of the points, therefore movable. All simulations I know of (beginning with those performed by IBM colleagues around 1960) suggest the following. Even if the pattern of the singularities begins by being uniform or Poisson, it gradually creates clusters and a semblance of hierarchy, and appears to tend toward fractality. It is against the preceding background that I conjectured that the limit distribution of galaxies is fractal, and that the origin of fractality lies in Newton's equations.

The Navier-Stokes and Euler equations of fluid motion and fractality of their singularities. *Background.* It is worth noting that the first concrete use of a Cantor dust in real spaces is found in a 1963 paper on noise records by Berger & Mandelbrot (reprinted in *SN*). This work was near simultaneous with Kolmogorov's work on the intermittence of turbulence. After numerous experimental tests, designed to create an intuitive feeling for this phenomenon (e.g., listening to turbulent velocity records that were made audible), I extended the fractal viewpoint to turbulence, and was led circa 1964 to the following conjecture.

Conjecture concerning facts. The property of being "turbulently dissipative" should not be viewed as attached to domains in a fluid with significant interior points, but as attached to fractal sets. In a first approximation, those sets' intersection with a straight line is a Cantor-like fractal dust having a dimension in the range from 0.5 to 0.6. The corresponding full sets in space should therefore be expected to be fractals with Hausdorff dimension in the range from 2.5 to 2.6.

Actually, Cantor dust and Hausdorff dimension are not the proper notions in the context of viscous fluids, because viscosity necessarily erases the fine detail that is essential to Cantor fractals. Hence the following.

Conjecture: (*FGN*, Chapter 11 and Mandelbrot, *C. R. Acad. Sci. Paris*: 282A, 1976, 119, translated as Chapter N19 of *SN*). The dissipation in a viscous fluid occurs in the neighborhood of a singularity of a nonviscous approximation following Euler's equations, and the motion of a nonviscous fluid acquires singularities that are sets of dimension about 2.5 to 2.6.

Open mathematical problem: To prove or disprove this conjecture, under suitable conditions.

Comment A. Several numerical tests agree with this conjecture (e.g., Chorin *Commun. Pure and Applied Math.*: 34, 1981, 853).

Comment B. I also conjectured that the Navier-Stokes equations have fractal singularities, of much smaller dimension. This conjecture has led to extensive work by V. Scheffer, R. Teman and C. Foias, and many others. But this topic is not exhausted.

Comment C. As is well known to students of chaos, a few years after my work, fractals in phase space entered the transition from laminar to turbulent flow, through the work of Ruelle and Takens and their followers. The task of unifying the real and phase-space roles of fractals is not completed.

9 Iterates of the complex map $z^2 + c$. Julia and Mandelbrot sets

The study of iterates of rational functions of a complex variable reached a peak circa 1918. Fatou and Julia succeeded so well that – apart from the proof of the existence of Siegel discs – their theory remained largely unchanged for sixty years.

The J set or Julia set. This set, defined as the repellor of rational iteration, is typically a fractal: a nonanalytic curve or a "Cantor-like" dust. Julia called these sets "very irregular and complicated." The computer – which I was the first to use systematically – reveals they are beautiful. To associate forever the name of Fatou and Julia, the complement of the Julia set is best called the Fatou set and its maximal open components, Fatou domains. The wildly colorful displays that represent them must now be familiar to every reader.

Starting with the quadratic map $z \to z^2 + c$, I explored numerically and graphically how the value of c affects the nature of quadratic dynamics, and in particular, the shape of the Julia set.

The M_0 set. Of greatest interest from the viewpoint of dynamics, hence of physics, is the set M_0 of those values of c for which $z^2 + c$ has a finite stable limit cycle.

The M_0 set having proved to be hard to investigate directly, I moved on to the computer-assisted investigation of a set that is easier to study, and seemed closely related.

The M set. The set of those parameter values c in the complex plane, for which the Julia set is connected, was called the μ-map in *FGN* (Chap. 19), but Douady and Hubbard called it *the Mandelbrot set*.

M proved to be a most worthy object of study, first for "experimental mathematics" and then for mathematics, and also for a new form of art! It is so well and so widely known, that no further reference is needed. But it is good to mention that the M set is a universal object. Curry, Garnett, and Sullivan (*Commun. Math. Phys.* 91, 1983, 267) discovered that the M set arises also in Newton's method for cubic polynomials, a dynamical system significantly different from $z \to z^2+c$. Following this, Douady and Hubbard (*Ann. Sci. Ec. Norm. Sup. (Paris)*: 18, 1985, 287) developed the theory of quadratic-like maps and showed that the M set arises for a wide variety of functions, and in this sense is a universal object.

Also, the study of $z \to z^2 + c$ naturally suggested the study of similar questions for other polynomials. But even the generic cubic, $z \to z^3 + az + b$, has proved soberingly difficult. Intense study by extremely powerful mathematicians still leaves many questions unanswered.

Conjecture that M is the closure of M_0. Computer approximations of M_0 actually represent a smaller set, and computer approximations of M actually represent a larger set. Extending the duration of the computation seemed to make the two representations converge to each other. Furthermore, when c is an interior point of M, not too close to the boundary, it was easily checked that a finite limit cycle exists. Those observations led to the conjecture that M is identical to M_0 together with its limits points.

In terms of its being simple and understandable without any special preparation, this conjecture comes close to where this paper starts: the "dimension 4/3" conjecture about tame Brownian motion. Again, I could think of no proof, even of a heuristic one. More significantly, after eighteen-odd years, the conjecture remains unanswered.

The MLC conjecture. Many equivalent statements were identified, the best known being that the Mandelbrot set is locally connected. This statement received the "nickname," MLC; it has the great advantage of being local (and J. C. Yoccoz proved it for a very large subset of the boundary). But, compared to the original form, it has the great drawback of being far from intuitive. (For the generic cubic, the corresponding local connectivity conjecture was proved to be false.)

10 Limit sets of Kleinian groups

A collection of Möbius transformations of the form $z \to (az+b)/(cz+d)$ defines a group that Poincaré called Kleinian. With few exceptions, their limit sets S are fractal. For the closely related groups based on geometric inversions in a collection $C_1, C_2, \ldots, C_n$ of circles, there is a well-known algorithm that yields S in the limit. But it converges with excruciating slowness as seen in Plate 173 of *FGN*. For a century, the challenge to obtain a fast algorithm remained unanswered, but it was met in many cases in Chapter 18 of *FGN*. (See also *Mathematical Intelligencer*: 5(2), 1983, 9.) In the case of this construction, fractal geometry did not open a new mathematical problem, but helped close *a very old* one.

In the new algorithm, the limit set of the group of transformations generated by inversions is specified by covering the complement of S by a denumerable collection of circles that "osculate" S. The circles' radii decrease rapidly, therefore their union outlines S very efficiently.

When S is a Jordan curve (as on Plate 177 of *FGN*), two collections of osculating circles outline S, respectively from the inside and the outside. They are closely reminiscent of the collection of osculating triangles that outline Koch's snowflake curve from both sides. Because of this analogy, the osculating construction seems, after the fact, to be very "natural." But the hundred year gap before it was discovered shows it was not obvious. It came only after respectful examination of pictures of many special examples.

A particularly striking example is seen in Figure 3, called "Pharaoh's breastplate," a rendering of Plate 199 of *FGN*. A more elaborate version of this picture appears on the cover of *SN*. This is the limit set of a group generated by inversion in the 6 circles drawn as thin lines on the small accompanying diagram. Here, the basic osculating circles actually belong to the limit set and do not intersect (each is the limit set of a Fuchsian subgroup based on three circles). The other osculating circles follow by all sequences of inversions in the 6 generators, meaning that each osculator generates a "clan" with its own color.

By inspection, it is easy to see this group also has three additional Fuchsian subgroups, each made of four generators and contributing full circles to the limit set.

Pictures such as Figure 3 are not only aesthetically pleasing, but they breathe new life into the study of Kleinian groups. Thurston's work on hyperbolic geometry and 3-manifolds opens up the possibility for limit sets of Kleinian group actions to play a role

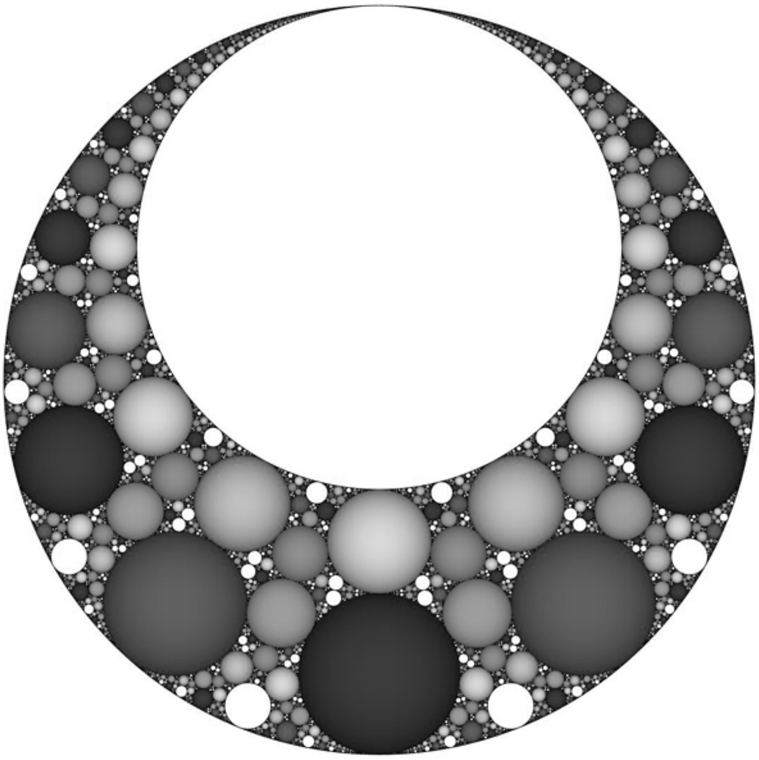

Figure 3

in the attempts to classify 3-manifolds. The Hausdorff dimension of these limit sets has been studied for some time by Sullivan, Canary, and others.

Challenge. Incorporate lacunarity and multifractal measures into the study of 3-manifolds through these limit sets.

11 Conclusion

The scope of this paper is necessarily limited. Many other fractal challenges and/or conjectures remain unanswered. Still others have been met and/or confirmed (especially in the context of multifractals). Among fields of research, fractal geometry may still prove to exhibit the shortest distances and the greatest contrasts between a straightforward core, which has by now become widely known, even to children and adult amateurs, and multiple new frontiers filled with major difficulties of every kind.

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General references in addition to specific references in the text.

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