3 A Geometry Able to Include Mountains and Clouds

Benoît Mandelbrot

This chapter originated in 'A Lecture on Fractals' delivered at a Nobel Conference at Gustavus Adolphus College in St Peter (Minnesota) in 1990. Mandelbrot's wide-ranging presentation and the tenor of his responses in the discussion following the lecture demonstrate the ubiquity of fractals, from nature to art and from economics to physics.

Benoît Mandelbrot was born in Poland in 1924, and moved with his family to Paris in the 1930s where one of his uncles introduced him to the Julia sets. Despite what he calls 'chaotic schooling', Mandelbrot obtained his Ph.D. in Paris in 1952. A few years later he moved to the US, and became an IBM Fellow (now Fellow Emeritus) at the Thomas B. Watson Research Center in New York. He pursued his intuitions about fractals by using the rare opportunity of massive computer power to test and prove his idiosyncratic ideas. After prestigious university appointments in a variety of subjects, he joined the Yale faculty in 1987, where he is Sterling Professor of Mathematical Science.

Mandelbrot is world-renowned for developing fractal geometry and discovering the Mandelbrot Set, named in his honour. He has written and lectured widely and has received numerous academic honours, including The Wolf Prize for Physics in 1993 and The Japan Prize for Science and Technology in 2003.

In order to understand geometric shapes, I believe that one must see them

It has very often been forgotten that geometry simply *must* have a visual component, and I believe that in many contexts this omission has proven to be very harmful.

To begin, let me say a few words concerning the scope of fractal geometry. In 1990, I saw it as a workable geometric middle ground between the excessive geometric order of Euclid and the geometric chaos of general mathematics. It is based on a form of symmetry that had previously been underutilized, namely self-similarity, or some more general form of invariance under contraction or dilation.

Fractal geometry is conveniently viewed as a language, and it has proved its value by its uses. Its uses in art and pure mathematics, being without practical application, can be said to be poetic. Its uses in various areas of the study of materials and other areas of engineering are examples of practical prose. Its uses in physical theory, especially in conjunction with the basic equations of mathematical physics, combine poetry and high prose. Several of the problems that fractal geometry tackles involve old mysteries, some of them already known to primitive man, others mentioned in the Bible and others familiar to every landscape artist.

To elaborate, let us provide a marvellous text that Galileo wrote at the dawn of science:

> Philosophy is written in this great book – I am speaking of the Universe – which is constantly offered for our contemplation, but which cannot be read until we have learned its language and have become familiar with the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometric forms, without which it is humanly impossible to understand a single word of it; without which one wanders in vain across a dark labyrinth. (Galileo Galilei: *Il Saggiatore*, 1623)

We all know that mechanics and calculus, therefore all of quantitative science, were built on these characters, and we all know that these characters belong to Euclidean geometry. In addition, we all agree with Galileo that this geometry is necessary to describe the world around us, beginning with the motion of planets and the fall of stones on Earth.

A geometry of nature?

But is it sufficient? To answer, let us focus on that part of the world that we see in everyday life. Modern box-like buildings are cubes or parallelepipeds. Good-quality plasterboard is flat. Good-quality tables are flat and typically have straight or circular edges. More generally, the works of Man, as the engineer and the builder, are typically flat, round or follow the other very simple shapes of the classical schools of geometry.

By contrast, many shapes of nature – for example, those shapes of mountains, clouds, broken stones, and trees – are far too complicated for Euclidean geometry. Mountains are not cones. Clouds are not spheres. Island coastlines are not circles. Rivers don't flow straight. Therefore, we must go beyond Euclid if we want to extend science to those aspects of nature.

A geometry able to include mountains and clouds now exists. I put it together in 1975, but of course it incorporates numerous pieces that have been around for a very long time. Like everything in science, this new geometry has very, very deep and long roots. Let me illustrate some of the tasks it can perform.

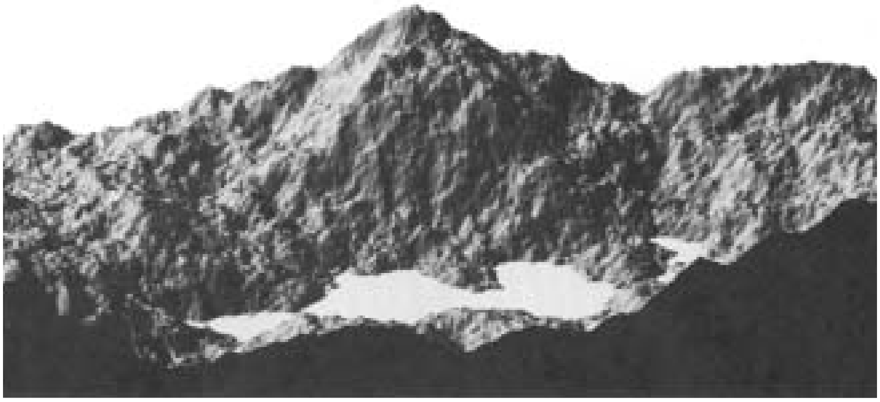


Fig. 3.1 left: A fractal landscape that never was (R.F. Voss)

Fig. 3.2 below: A cloud formation that never was (S. Lovejoy & B.B. Mandelbrot)

Figure 3.1 seems to represent a real mountain but is neither a photograph nor a painting. It is a computer forgery; it is completely based upon a mathematical formula from fractal geometry. The same is true of the forgery of a cloud that is shown in Fig. 3.2.

An amusing and important feature of these figures is that both adopt and adapt formulas that had been known in pure mathematics. Thanks to fractal geometry, diverse mathematical objects, which used to be viewed as being very far from physics, have turned out to be the proper tools for studying nature. I shall return to this in a moment.

Fractal modelling of relief was successful in an unexpected way. It is used in an immortal masterpiece of cinematography called *Star Trek Two, The Wrath of Khan*. Many people have seen it, but – unless prodded – few have noticed that the new planet that appears in the Genesis sequence of that film has a fractal relief. If I could show it to you, you would see that it happens to have peculiar characteristics (superhighways and square fields). They occur because of a shortcut taken by Lucasfilm in order to make it possible to compute these fractals quickly enough. But we need not dwell on flaws. Far more interesting is the fact that the films that include fractals create a bridge between two activities that are not expected to ever meet – mathematics and physics on the one hand, and popular art on the other.

More generally, fractals have an aspect that I found very surprising at the beginning and that continues to be a source of marvel: people respond to fractals in a deeply emotional fashion. They either like them or dislike them, but in either case the emotion is completely at variance with the boredom that most people feel towards classical geometry.

Let me state that I will never say anything negative about Euclid's geometry. I love it as it was an important part of my life as a child and as a student; in fact, the main reason why I survived academically, despite a chaotic schooling, was my geometric intuition, which allowed me to cover my lack of skill as a manipulator of formulae. But we all know by experience that, apart from professional geometers, almost everybody views Euclid as being cold and dry. The fractal shapes I am showing are exactly as geometric as those of Euclid, yet they evoke emotions that geometry is not expected or supposed to evoke.

The shape of deterministic chaos

Only one new geometry

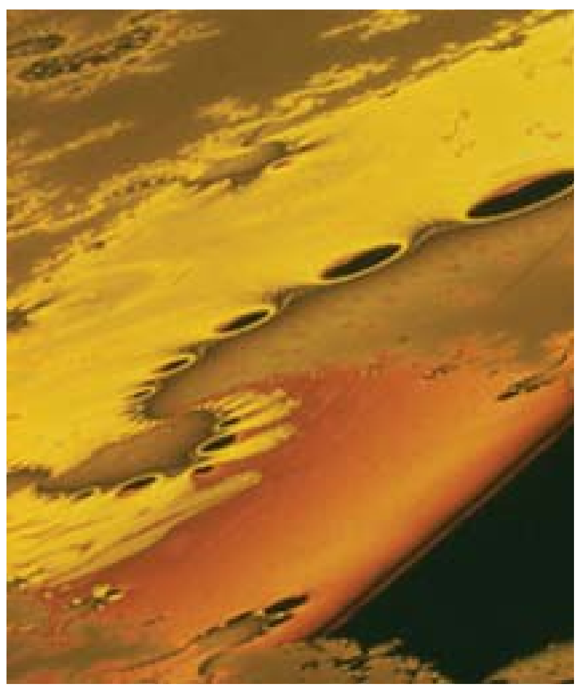
Now a few preliminary words about deterministic chaos. This topic will be touched on below, but something should be mentioned immediately. The proper geometry of deterministic chaos is the same as the proper geometry of the mountains and the clouds. Not only is fractal geometry the proper language to describe the shape of mountains and clouds, but it is also the proper language for all the geometric aspects of chaos. The fact that we need only one new geometry is really quite marvellous, because several might have been needed, in addition to that of Euclid.

I have myself devoted much effort to the study of deterministic chaos, and would like to show you now a few examples of the shapes I have encountered in this context.

Figure 3.3 is an enormously magnified fragment from a set to which my name has been attached. Here, a fragment has been magnified in a ratio equal to Avogadro's number, which is 10^{23}. Why choose this particular number? Because it's nice and very large, and such a huge magnification provides a good opportunity for testing the quadruple-precision arithmetic on the IBM computers of a few years ago. (They passed the test. It's very amusing to be able to justify plain fun and pure science on the basis of such down-to-earth specific jobs.) If the whole Mandelbrot Set had been drawn on the same scale, the end of it would be somewhere near the star Sirius.

Fig. 3.3 above: A very small fragment of the Mandelbrot Set (R.F. Voss)

Fig. 3.4 below: A small fragment of a modified Mandelbrot Set (B.B. Mandelbrot)



The shape of the black bug near the centre is very nearly the same as that of the centre of the whole Mandelbrot Set, to be discussed later when I return to this topic. Finding bugs all over is a token of geometric orderliness. On the other hand, the surrounding patterns vary from bug to bug. This is a token of variety.

The shape shown in Fig. 3.4 is a variant of the Mandelbrot Set that corresponds to a slightly different formula. This shape is reproduced here simply to comment on a totally amazing and extraordinarily satisfying aspect of fractal geometry. Fractals are perceived by many people as being beautiful, but were initially developed for the purpose of science, for the purpose of understanding how the world is put together – both statically (in terms of mountains) and dynamically (in terms of chaos, strange extractors, etc.).

In other words, the shapes shown in Figs. 3.1 to 3.4 were not *intended* to be beautiful. So why is it that they are *perceived* as beautiful? The fact that they are must tell us about something regarding our system of visual perception.

I started with these four figures because their structure is so rich, but I went overboard. The richness of their structure means that these figures cannot be used to explain the main feature of all fractals. The underlying basic principle shows far more clearly on Fig. 3.5, which – for a change – reproduces a real photograph of a real object. You may recognize the *Romanesco* variety of cauliflower. Each bud looks absolutely like the whole head, and in turn, each bud subdivides into smaller buds, and so on. I am told that the same structure repeats over five levels of separation that you can see with the naked eye, and then through many more levels that you can only see with a magnifying glass or microscope.

Scientists' first reaction to such shapes was to focus on the spirals formed by the buds. This interest

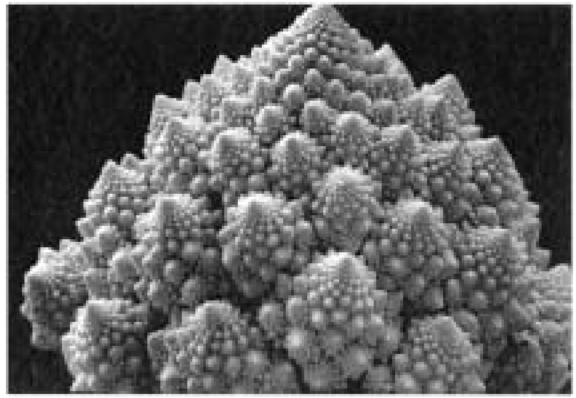


Fig. 3.5 above: Cauliflower *Romanesco*

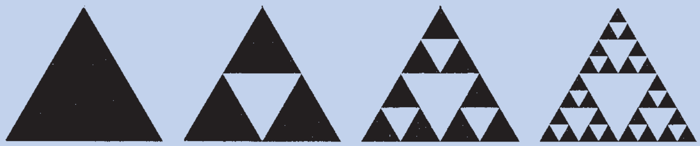
led to extensive knowledge about the relation between the golden mean (and the Fibonacci series), and the way plants spiral. But to me what is more important is the hierarchical structure of buds because it embodies the essential idea behind fractals.

What is a fractal ?

Before we go on to tackle what a fractal *is*, let us ponder what a fractal *is not*. Zoom on to a geometric shape and examine it in increasing detail. That is, take smaller and smaller portions near a point P, and allow every one to be dilated, that is, enlarged to some prescribed overall size.

If our shape belongs to standard geometry, it is well known that the enlargements become increasingly smooth. That is, one expects a curve to be 'attracted', under dilations, towards a straight line (thus defining the tangent at the point P). The term 'attractor' is borrowed from dynamics and probability theory. One also expects a curve to be attracted under dilation to a plane (thus defining the tangent plane at the point P).

An exception to this rule is when P is a double point of a curve; the curve near P is then attracted to



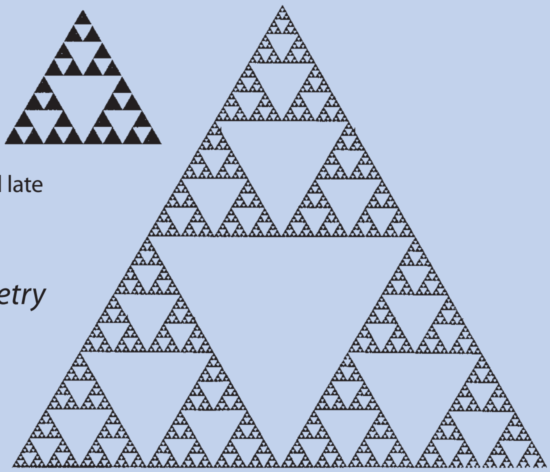


Fig. 3.6 above and right: The Sierpinski gasket: early and late stages of construction

The fact that we need only one new geometry is really quite marvellous, because several might have been needed, in addition to that of Euclid.

two intersecting lines and has two tangents, but double points are few and far between in standard curves. In general, one can say that nearly every standard shape's local structure converges under dilation to one of the small number of 'universal attractors'. The grandiose term universal is borrowed from recent physics.

Yet the shapes I have been showing fail to be locally linear. In fact, they deserve to be called 'geometrically chaotic' until proven otherwise. In an isolated neighbourhood of the great City of Science, a kind of geometric chaos was discovered in the fifty years from 1875. Then, while trying to escape their concern about nature, mathematicians became aware of the fact that a geometric shape's roughness *need not* vanish as the examination becomes more searching. It is conceivable that it should either remain constant, or endlessly vary up and down.

The hold of standard geometry was so powerful, however, that the resulting shapes were not recognized as models of nature. Quite to the contrary, their discoverer proudly labelled them 'monstrous' and 'pathological'. After discovering these sets, mathematics proceeded to increasingly greater generality.

Like a sailor, science must constantly navigate between two dangers: the lack of and excess of generality. Between the extremes of the excessive geometric order of Euclid, and of the geometric chaos of the most general mathematics, can there be a middle ground? To provide one is the ambition of fractal geometry.

The essential nature of fractals

The reason why fractals are far more special than the most general shapes of mathematics, is because they are characterized by so-called 'symmetries', which are invariances under dilations and/or contractions. Broadly speaking, mathematical and natural fractals are shapes whose roughness and fragmentation *neither* tend to vanish, *nor* fluctuate up and down, but remain *essentially unchanged* as one continually zooms in. Hence, the structure of every piece holds the key to the whole structure.

The preceding statement is made precise and illustrated by Fig. 3.6, which represents a shape that is enormously more simple than those shown previously. As a joke, I called it the 'Sierpinski gasket', and the joke has stuck.

The four small diagrams show the 'initiator' of the construction, which is a triangle, then its first three stages, while the large diagram shows an advanced stage. The basic step of the construction is to divide a given (black) triangle into four sub-triangles, and then erase (whiten) the middle fourth. This step is first performed with a wholly black filled-in triangle of side 1, then with three remaining black triangles of side $^1/_2$. This process continues, following a pattern called *recursive deletion*, which is very widely used to construct fractals. Related patterns are *recursive substitution* and *recursive addition* (which we shall encounter) and *recursive multiplication* (which is fundamental but beyond the scope of this talk).

Now, take the gasket and perform an isotropic linear reduction whose ratio is the same in all directions – namely $^1/_2$ – and whose fixed point is any of the three apexes of the initiator triangle. This transformation is called a *similarity*. More precisely, it is *homothety* or *linear* self-similarity. By examining the large advanced stage picture, it is obvious that each of the three reduced gaskets is simply superposed on one-third of the overall shape. For this reason, the fractal gasket is said to have three properties of *self-similarity*.

The essence of self-similarity

Precise terminology is necessary here because one can also understand 'similar' as a loose everyday synonym of 'analogous'. In the early days of fractal geometry, the resulting terminological ambiguity was acceptable to physicists, because early detailed studies did indeed concentrate on linearly self-similar shapes. However, later developments have extended to *self-affine* shapes, in which the reductions are still linear, but the reduction ratios in different directions are different. For example, in order to go from a large to a small piece of fractal relief, one must contract the horizontal and vertical coordinates in different ratios. Hence, a fractal relief is called linearly self-affine.

When the Sierpinski gasket is constructed by deleting middle triangles, as in Fig. 3.6, its self-similarity seems, so to speak, to be 'static' and 'after-the-fact'. But this is a completely misleading impression. Its prevalence and its being viewed as a flaw are continual sources of surprise. In fact, the same symmetries can be reinterpreted 'dynamically' and suffice to generate the gasket. The device, which is called the 'chaos game', is a stochastic, or randomly determined interpretation of a scheme made by Hutchinson. Start with an 'initiator', that is, an arbitrary bounded set, for example a P_0. Denote the three similarities of the gasket by S_0, S_1, and S_2, and denote by $k(m)$ a random sequence of the digits 0, 1 and 2. Then define an 'orbit', as made of the points $P_1 = S_{k(1)}(P_0)$, $P_2 = S_{k(2)}(P_1)$ and more generally $P_j = S_{k(j)}(P_{j-1})$. One finds that this orbit is 'attracted' to the gasket, and that after a few stages it describes its shape very well.

In 1964, when I first used the word 'self-similarity', I thought it was a neologism. In fact at least one writer had used it before. But the idea itself is perfectly obvious and must be very old. The reason the word was needed is that the shapes to which it refers had no importance until my work. For example, Sierpinski had defined his shape for some purpose that has long been forgotten – because it was not very important.

Why did self-similarity become important? Because Figs. 3.1 to 3.5 are self-similar, not – to be sure – in an exact, but in a slightly loose meaning of the word. Why fractal geometry has become such a large subject, and why I spent so much time in my efforts to build it as a discipline, is driven by the empirical discoveries (each established by a separate investigation) that the relief of planet Earth is self-similar, and that the same is true of many other shapes around us. Figures 3.1 to 3.5 suffice to show that the impression that self-similarity is a barren and not very fruitful idea would be an altogether wrong impression.

Granted what has just been asserted, why did the gasket become important? It does not represent anything of interest; in fact, it is so relentlessly monotonous that it could be seen as being as simple as Euclid. You can know nearly everything about it in just a few days of study. The same holds for another widely known shape, called the snowflake curve or Von Koch Island, for a set Cantor Dust, and for a few other long-known structures of the same ilk. The reason why they are important is because you must begin the study of fractal geometry *with* the Sierpinski gasket and its type, but keep in mind that the real fun begins *beyond* them.

The new Peano curve

The fun begins after one has added an element of unpredictability, due to either randomness (as in Figs. 3.1, 3.2 and 3.5) or non-linearity (as in Figs. 3.3 and 3.5). Non-linearity is the key word of the new meaning of chaos, namely of deterministic chaos, and randomness is the key to chaos in the old sense of the word. The two are very intimately linked.

But let us not rush away from linearly self-similar fractals, because in some cases a suitable graphic rendering suffices to break their relentless monotony.

Figure 3.7 shows my variant of a curve that Giuseppe Peano constructed in 1890. The point of Peano curves is that they manage to fill a portion of the plane, hence contradict the basis of the notion of curves. Mathematicians have written pages and pages to praise the freedom of imagination that allows man to invent shapes that are completely removed from reality. The Peano curve was specifically designed to be a counterexample to a natural belief that used to be universal: that curves and surfaces do not mix. It was designed for the purpose of separating mathematics and physics into two completely independent investigations. Unfortunately, it was quite successful in that respect, at least for a century.

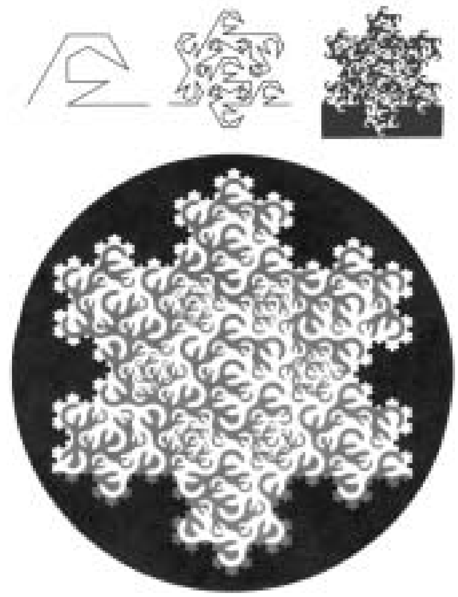


Fig. 3.7 above: Mandelbrot's Peano curve (B.B. Mandelbrot)

To obtain my new Peano curve, you replace an initial straight segment by the complicated zigzag (top left). Then (top middle) each zig and zag is replaced by smaller versions of the zigzag on the top left. The same pattern (called *recursive substitution*) is then repeated without end. In the top-right diagram, it is easy to believe that the boundary between black and white will end up filling a snowflake curve. I call it a 'snowflake sweep'. The bottom of Fig. 3.7 reproduces the same curve but will replace every segment by an arc of a circle.

This fancy computer rendering was great fun but had a very practical goal. It was carefully thought through to *force* everybody to see all kinds of branching systems of arteries and veins, or of rivers, or of flames or whatever else you prefer. But those very realistic things were not seen until my work, if only because mathematicians spurned their ability to see. Partly as a result, mathematics and physics did indeed move in very different directions.

Figure 3.8 combines a sequence of completely

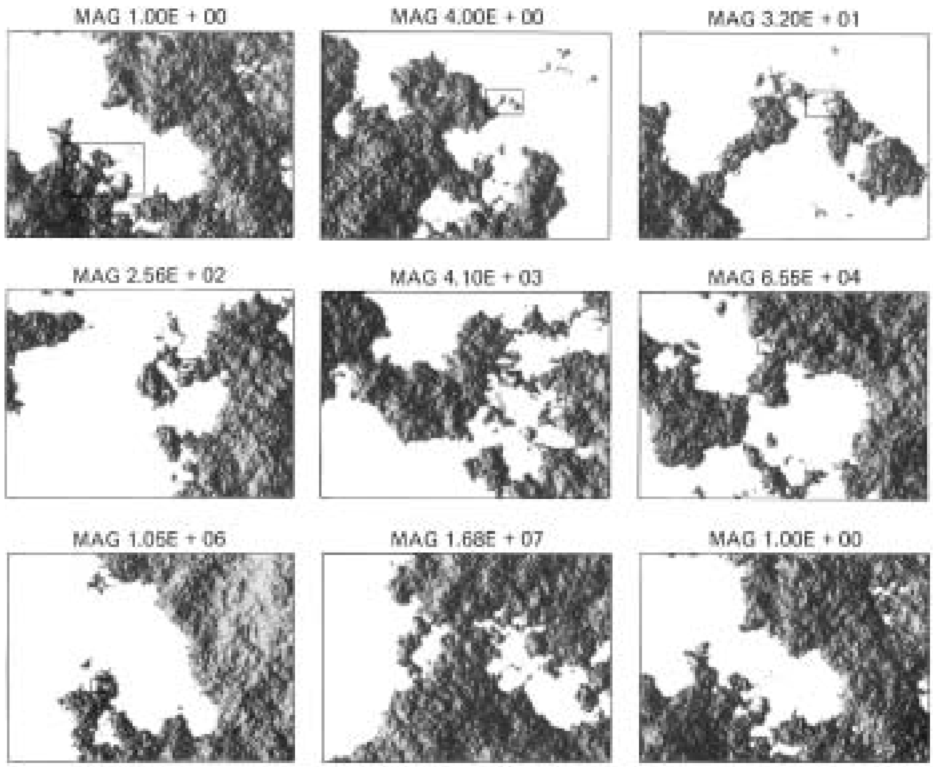
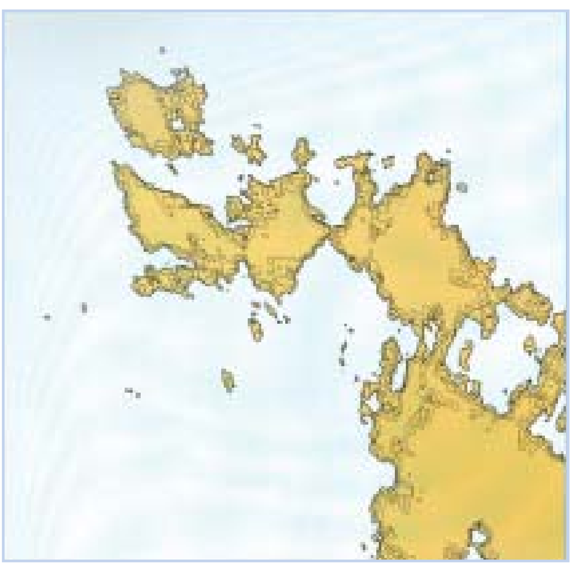


Fig. 3.8 right: Zoom onto a fractal landscape that never was (R.F. Voss)

Fig. 3.9 below: A fractal coastline that never was (B.B. Mandelbrot)



artificial, random landscapes. Each part of this picture consists of enlarging a small black rectangle in the preceding picture and then filling in additional detail. This procedure is called *recursive addition*. Each landscape differs from the preceding one by being more detailed, yet at the same time the successive enlargements are comparable. They might have been different parts of the same coastline examined on the same scale, but in fact they are neighbourhoods of one single point examined at very different scales. Clearly, these successive enlargements of a coastline completely fail to converge to a limit tangent!

How to measure roughness

At this point, let me recall a story about the great difficulties the ancient Greeks used to experience in formulating the idea of 'size'. Navigators knew that Sardinia took longer to circumnavigate than Sicily. On the other hand, there was evidence that Sardinia's fields are smaller than Sicily's. So which was the bigger island? Greeks sailors seem to have long held the belief that Sardinia was the bigger of the two, because its coastline was longer.

But let us examine Fig. 3.9, and ponder the notion of coastline length. When the ship used to circumnavigate is large, the captain will report a rather small length. A much smaller ship would come closer to the shore and navigate along a longer curve. A man walking along the coastline will measure an even longer length. So what about the 'real length of the coast of Sardinia'? The question seems both elementary and silly, but it turns out to have an unexpected answer. The answer is, 'it depends'. The length of a coastline depends on whether you circumnavigate it in a large or a small ship, or walk along it, or use a mouse or some other instrument to measure the coastline.

This makes us appreciate the extraordinary power of the mental structure that schools have imposed by restricting their teaching of geometry to Euclid. Many people thought they never understood geometry, yet they learned enough to expect every curve to have a length. For the curves in which I am interested, this turns out to have been the wrong thing to remember from school. Once again, the theoretical length is infinite, and the practical length depends on the method of measurement. Its increase is faster where the coastline is rough, making it necessary to study the notion of roughness.

This last notion is fundamental, because the world we live in includes many rough objects that can cause great harm. Man must learn to live among those objects. However, the task of measuring roughness objectively has turned out to be extraordinarily difficult. People whose work demands it, like metallurgists, merely went to their friends in statistics asking for a number they could measure and call roughness.

But the following experiment reveals a serious problem. Take samples of steel that the US National Bureau of Standards guarantees to come from one block of metal as as homogeneous as man can make it. Break the steel samples and measure the roughness of the fractures, evaluated according to the rules of statistics. You will find that the values you get are in complete disagreement.

So what about the 'real length of the coast of Sardinia'? The question seems both elementary and silly, but it turns out to have an unexpected answer…it depends on whether you circumnavigate it in a ..ship…or walk along it, or use a mouse or some other instrument to measure the coastline.

Fractal dimension is the answer

On the other hand, I argue that roughness happens to be measured consistently by a quantity called *fractal dimension*, which happens in general to be a fraction, and which one can measure very accurately. Studying many samples from the same block of metal, we found the same dimension for every sample.

The idea is that fractal dimension is a proper measure for the notion of roughness just as temperature is a proper measure for the notion of hotness. Man must have known forever that some things are hot and others are cold, but before physics could move on to a theory of matter, it was necessary to describe the degree of hotness by one number. This was possible only when the thermometer was invented, and different people using the same thermometer could get the same value of hotness for the same object.

Similarly and most fortunately, fractal geometry started with a few ideas about how to express roughness and complexity by a number. Some of these ideas add up a bunch of related but distinct tools (one can think of them as being different types of screwdrivers) that are collectively called 'fractal dimensions'. People who work with fractal geometry quickly develop an intuition of fractal dimension and can now guess it very accurately for simple shapes.

The reason we use the term 'dimension' is that it can also be applied to points, intervals, full squares and full cubes, and in those cases yield the familiar values of 0, 1, 2 and 3. Applied to fractals, however, these definitions usually yield values that are not integers. The loose idea of 'roughness' has turned out to demand a number of distinct numerical implementations, hence the multiplicity of distinct 'fractal dimensions' has proven valuable. A dimension delineated by Hausdorff and Besicovitch was the first example, but for practical needs it is either too difficult or too specialized.

The simplest variant is the *similarity* dimension D_s, which applies to shapes that are linearly self-similar. As I have already stated, this means that they are made up of N replicas of the whole, each replica being reduced linearly in the same ratio r. Then one defines

$$D_s = \frac{\log N}{\log(1/r)}$$

For a point, an interval, a square and a full cube, one has $D_s = 0, 1, 2$ and 3, respectively. As announced, these are the familiar values of the 'ordinary' dimensions. But the Sierpinski gasket adds something very new: one has $N = 3$ and $r = 1/2$, hence $D = {}^{\log 3}/_{\log 2} \sim 1.5849 \ldots$

Another simple fractal dimension is the *mass dimension*. Take a distribution of mass of uniform density on the line, in the place or in space. Then choose a sphere of radius R whose centre lies in our set. The mass in such a sphere takes the form $MR = FR^D$, where D is the 'ordinary' dimension and F is a numerical constant. The idea of uniform density extends to fractals, and in many cases an exponent D can be defined; it is called the *mass dimension* and is often equal to the similarity dimension. Unfortunately, we must move away from dimension.

How to grow a tree

The next subject I wish to tackle is the increasingly valuable role of fractal geometry as tool in the discovery and study of previously unknown aspects of nature. Nothing illustrates this role better than a form of random growth that generates the Fractal *Diffusion Limited Aggregates* (DLA) or Witten-Sander aggregates. A DLA cluster lurks in the centre of Fig. 3.10. It is a tree-like shape of baffling complexity that one can use to model how ash forms, how water seeps through rock, how cracks spread in a solid and how lightning discharges.

To see how the growth proceeds, take a very large chess board and place a queen that is not allowed to move in the central square. Pawns are allowed to move in any of the four directions on the board. They are released from a random starting point at the edge of the board, and are instructed to perform a random or drunkard's walk. Each step can take one of four directions chosen with equal probabilities. When a pawn reaches a square next to that of the original queen, it transforms itself into a new queen and cannot move any further. Eventually, one has a branched, spidery collection of queens.

Quite unexpectedly, massive computer simulations show that DLA clusters are fractal. They are nearly self-similar, that is, small portions are very much like reduced versions of large portions. But deviations from randomized linear self-similarity are obvious and pose interesting challenges.

One reason for the importance of DLA is that it concerns the interface between the smooth and the fractal. A premise of fractal geometry is that much in the world is fractal. Nevertheless, science is expected to be cumulative, the new being added to the old, without chasing it away. Therefore, new wisdoms must not deny the old wisdom that the world is made of smooth shapes and involves smooth variation and differential equations.

What DLA shows is that the old and new

wisdoms are compatible only if one abandons the old philosophical expectation that *everything* in the world will eventually prove to be smooth or of smooth variation.

To show how smooth variation can produce rugged behaviour, the original construction must first be rephrased in terms of the theory of electrostatic potential. The description that follows is necessarily a little schematic. Grow DLA in the big box connected to a positive potential (to be taken as unity) and connect the cluster itself to the potential 0. Then the value of the potential elsewhere in the box is best described by equipotential curves, for example, the curves along which the potential takes the increasing values .01, .02, … .99.

Figure 3.10 shows that all these curves are smooth and that they provide a progressive transition between the box and the boundary of the cluster. Analytic calculation is out of the question, but 'physical common sense' can be combined with numerical calculation. In effect, the object's boundary includes many needles, and each has a high probability of getting hit by lightning. This is manifested by the fact that equipotential lines crowd together near the tips of a DLA cluster. More generally, returning to the random pawns that build up a DLA cluster, the position where the pawn lands is obtained from the shapes of the electrostatic equipotentials.

Now we come to the next logical step, which implies that DLA has brought an intellectual innovation of the highest order. For nearly 200 years, the study of potentials has limited itself to fixed boundaries. But in the simple random walk that creates DLA, a 'hit' in the above terminology can be interpreted as provoking a displacement of the boundary. Thus, the massive numerical experiments about DLA teach us that when one allows boundaries to move in response to the potential, the boundaries become fractal.

This shows without any trace of doubt that one can create rough fractals from the smoothness that characterizes equipotential lines, but this knowledge remains imperfect. We all thirst for new mathematics and physics. Nevertheless, it is worth noting how fractal geometry has led to an altogether new problem, outlined the broad solution and set many scientists to work.

The Julia Set

Our next move returns from randomness to deterministic chaos, and replaces objects in real physical space by imaginary objects. What will remain unchanged is that we shall deal with spiky sets surrounded by smooth equipotential lines.

The first notion here is that of the Julia Set of quadratic iteration. Pick a point c of coordinates u and v, and call it a 'parameter'. Next, in a different plane, a point P_0 of the coordinates x_0 and y_0. Then form $x_1 = x_0^2 - y_0^2 + u$ and $y_1 = 2x_0y_0 + v$. These formulas may seem a bit artificial, but in order to satisfy the reader who is scared of complex numbers, they simplify if the point c of coordinates x and y is represented by the complex number $z = x + iy$. (Complex numbers add and multiply like ordinary numbers, except that i^2 must always be replaced by –1.) In terms of the complex numbers $c = u + iv$ and $z = x + iy$, the preceding rule simplifies to $z_1 = z_0^2 + c$ and (more generally) $z_{k+1} = z_k^2 + c$. Even the reader who is scared of complex numbers is able to understand the expressions in terms of x_k and y_k.

When the orbit P_k fails to escape to infinity, the initial P_0 is said to belong to the 'filled-in Julia Set'. An example is shown in Fig. 3.11. If you start outside of the black shape, you go to infinity. If you start inside, you fail to iterate to infinity.

The boundary between black and white is called a 'Julia curve'. It is approximately self-similar. Each chunk is not quite identical to a bigger chunk, because of non-linear deformation. Yet, it is aston-

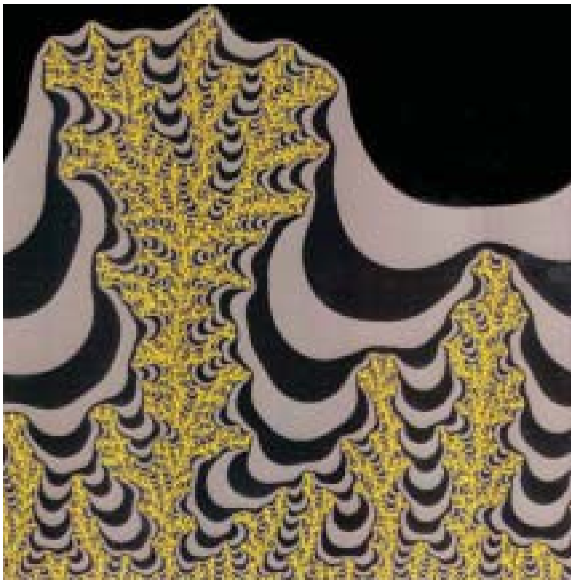


Fig. 3.10 above: A cluster of diffusion limited aggregation, surrounded by its equipotential curves (C.J.G. Evertsz and B.B. Mandelbrot)

ishing that iteration should create any form of self-similarity, quite spontaneously.

As in the investigation of fractal mountains, the computer was essential to the study of iteration. The bulk of fractal geometry is concerned with shapes of great apparent complication and they could never be drawn by hand. More precisely, this picture might have been computed by a hundred different people working for years, but nobody would have started such an enormous calculation without first feeling that it was worth performing.

Not only did I have access to a computer in 1979, but I was familiar with its power. The fact that no one knew what was going to emerge was enough to make these calculations worth trying. A fishing expedition led to a primitive form of Fig. 3.12. The Julia Sets of the map $z^2 + c$ can take all kinds of shapes, and a small change in C can change the Julia Set very greatly. I set out to classify all the possible shapes (for reasons that are too lengthy to discuss) and came up with a new shape. That it has been called the *Mandelbrot Set* is of course a great honour. Figure 3.3 above was a tiny portion of Fig. 3.12.

Constructing the Mandelbrot Set

Here is how the Mandelbrot Set is constructed. Take a starting point C_0 in the plane of coordinates u_0 and v_0. From the coordinate of C_0, form a second point C_1 of coordinates $u_1 = u_0^2 - v_0^2 + u_0$

and $v_1 = 2u_0u_0 + v_0$.

Next, form the point C_2 of coordinates

$u_2 = u_1^2 - v_1^2 + u_0$

and $v_2 = 2u_1u_1 + v_0$.

More generally, the coordinates u_k and v_k of C_k are obtained from u_{k-1} and v_{k-1} by the so-called 'iterative formulas'

$u_k = u_{k-1}^2 - v_{k-1}^2 + u_0$ and

$v_k = 2u_{k-1}v_{k-1} + v_0$.

When C_0 is represented by $z_0 = u_0 + iv_0$, the above formulas simplify to $z_1 = z_0^2 + z_0$ and $z_k = z_{k-1}^2 + z_0$. The points C_k are said to form the orbit of C_0, and the set M is defined as follows: If the orbit C_k fails to go to infinity, one says that C_0 is contained within the set M. If the orbit C_k does go to infinity, one says that the point C_0 is outside M.

This algorithm concerns the following very sober problem of deterministic dynamics. When C_0 is in the interior of M, quadratic dynamics yields an orbit that is perfectly orderly, in the sense that it is asymptotically periodic. When C_0 is outside M, to the contrary, the behaviour of the orbit is deterministic, but almost unpredictably, chaotic. Quadratic dynamics was singled out for detailed study because in this case the criterion separating orderly from chaotic behaviour is as clean as can be, as seen above. The boundary between the two possibilities turns out to be messy beyond any expectation.

Zooming towards a portion of the boundary of the Mandelbrot Set, you see two distinct phenomena. The part is simply a repetition of something already seen. This element of repetition is essential to beauty. But beauty also requires an element of change, and this is also very clearly present. As you come closer and closer, what you see becomes more and more complicated. The overall shape is the same, but the hair structure becomes more and more intense. This feature is *not* something we put in on purpose. In so far as the mathematics is concerned, it is not invented, but discovered: we see something that has been there forever. What we discover is that the mathematics of *z* squared plus C is astonishingly complicated, by contrast with the simplicity of the formula. We find that the Mandelbrot Set, when examined more and more closely, exhibits the co-existence of something that repeats itself relentlessly, something that exhibits a variety that boggles the imagination. I first saw the Mandelbrot Set on a black and white screen of very low graphic quality, and the picture looked dirty. But zooming in on what seemed like dirt revealed an extraordinary little copy of the whole.

In Fig. 3.12, the Mandelbrot Set is the *white* 'bug' in the middle. It is very rough-edged, but is surrounded by a collection of zebra stripes whose edges become increasingly smooth as one goes away from M. These zebra-stripe edges happen to be Laplacian equipotential curves. They are just like those in Fig. 3.10 but are far easier to obtain.

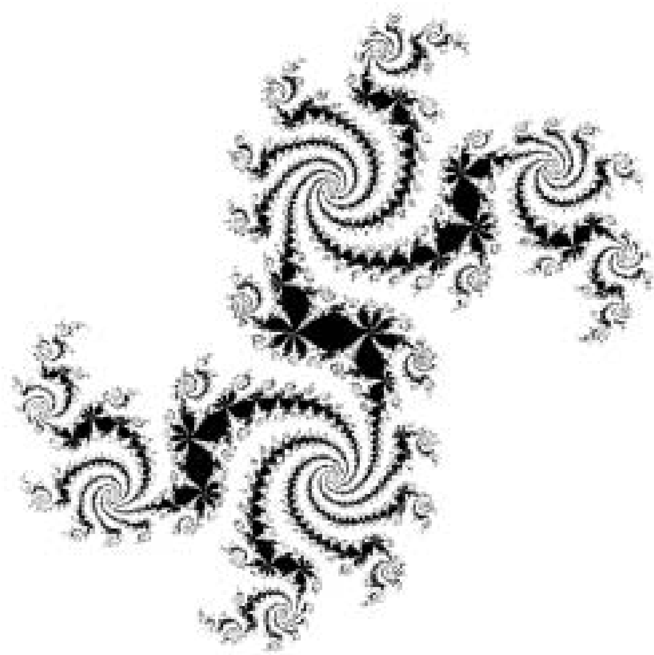
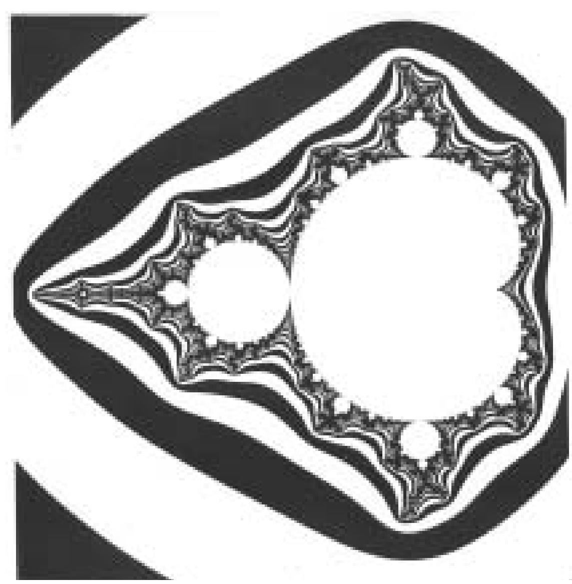


Fig. 3.11 above: Quadratic Julia Set for the map $z \rightleftharpoons z^2 + C$.

Fig. 3.12 below: The Mandelbrot Set, surrounded by its equipotential curves



Fractal art and the mathematician

To the layman, fractal art tends to seem simply magical, but no mathematician can fail to try to understand its structure and meaning. A remarkable aspect of recent events is that the mathematics triggered by the Mandelbrot Set could have passed as 'pure' if only its visual origin could have been hidden. To many mathematicians, the newly opened possibility of playing with pictures interactively has revealed a new mine of purely mathematical questions and conjectures, of isolated problems and whole theories. To take an example, examination of the Mandelbrot Set led me in 1980 to many conjectures that were simple to state, but then proved very hard to crack. (The main one remains unsolved.) To mathematicians, their being difficult and slow to develop does not make them any less fascinating, because a host of intrinsically interesting side-results have been obtained in their study.

Herein lies a tale. Pure mathematics is certainly one of the remarkable activities of man; it certainly is different in spirit from the art of creating pictures by numerical manipulation, and it has indeed proven that it can thrive in splendid isolation – at least over some brief periods. Nevertheless, the interaction between art, mathematics and fractals confirms what is suggested by almost all earlier experiences. Over the long haul, mathematics gains by not attempting to destroy the organic unity that appears to exist between seemingly disparate but equally worthy activities of man, the abstract and the intuitive.

Of course, the black and white figures in this chapter are not beautiful colour fractal pictures. As in the case of the mountains, the quality of the colour rendering shows the skills of the programmers, but the structure itself is independent of the colour rendering. What is important is that the structure is too complicated to be understood unless the colour rendering is sufficiently rich. In fact, the set has such an enormous amount of structure that we cannot see it in one single colour rendering. Different renderings emphasize very different aspects of it. Again, this structure was *not* invented for the purpose of doing something beautiful, but purely for the purpose of exploring the advanced theory of z squared plus C.

Simplicity generates marvellous complexity

Let me now bring together the separate strings of my chapter. How did fractals come to play their role of 'extracting order out of chaos'? The key resides in a very surprising discovery that I made thanks to computer graphics.

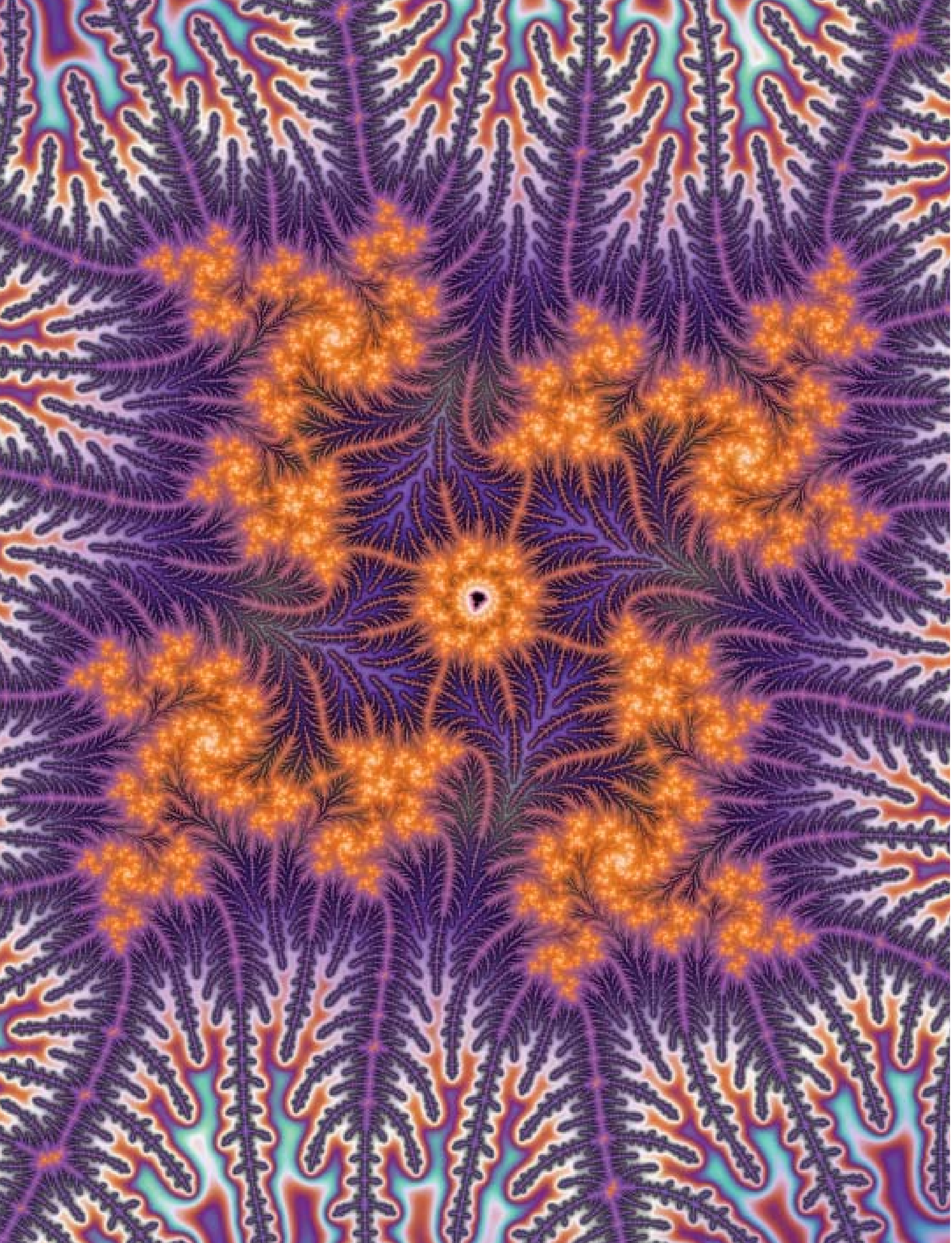
The algorithms that generate fractals are typically so extraordinarily short as to look positively dumb. This means they must be called 'simple'. Their fractal outputs, on the contrary, often appear to involve structures of great richness. A priori, one would expect the construction of complex shapes to necessitate complex rules, but surprisingly, it is not so.

What is the special feature that makes fractal geometry perform in such an unusual manner? The answer is very simple. The algorithms are recursive, and the computer code written to represent them involves 'loops'. That is, the basic instructions are simple, and their effects can be followed easily.

Let these simple instructions be followed repeatedly. Unless one deals with the simple old fractals (the Cantor Set and Sierpinski gasket), the process of iteration effectively builds up an increasingly complicated transform, whose effects the mind can follow less and less easily. Eventually, one reaches something that is qualitatively different from the original building block. One can say that the situation is a fulfilment of what in general is nothing but a dream: the hope of describing and explaining chaotic nature as the cumulation of many simple steps.

Many fractals have been accepted as works in a new form of art. Some are representational, while others are totally unreal and abstract, yet all strike almost everyone in forceful, almost sensual, fashion. The artist, the child and the 'man in the street' can never see enough as they never expect to get anything of this sort from mathematics.

Nor did mathematicians expect their subject to interact with art in this way. Eugene Wigner has written about 'the unreasonable effectiveness of mathematics in the natural sciences'. To this line, I have been privileged to add another parallel statement, concerning 'the unreasonable effectiveness of mathematics as creator of shapes that Man can marvel about, and enjoy'.



After Benoît Mandelbrot had delivered this paper, he answered some questions:

Chairman: First of all, are there any responses from the panel?

Q1: Benoît, if I could ask, speaking of poetry and prose, this is a rather flippant question, but is it more like music or like noise?

Mandelbrot: For me, music is a form of poetry, and I forgot to say so simply because I felt it was obvious. Analogies can become very dangerous if pursued too far, but I'm glad that you have been taken by the game.

Q2: Benoît has given us a lot of very nice insights, I think, into this kind of geometry, but I would like to express a little, maybe different point of view, which emphasizes something else. I think Benoît spoke at one point, something to the effect that the physical properties reduced to the geometric properties. And I think, somehow the geometry is very static, and to me the static should best be seen with deeper understanding as flowing from the dynamics. Therefore, I would put a dynamical perspective on the understanding of physics, above that of a geometrical perspective. In the dynamics, the physical process itself, the equations, which are time-dependent, from those one can derive some of these fractal geometric pictures with a deeper understanding than just looking at the pure fractal geometry in its own right. So, for me, there's a little more primary emphasis on the deeper physics coming from the dynamics rather than the geometry.

Mandelbrot: I see absolutely no conflict between our viewpoints. To study the dynamics of Julia Sets, you must study the statics of the Mandelbrot Set. In many cases, for example, the shape of the mountains, everyone knows well, is static. If so, the next step would be to understand the processes that create the mountains. This task is far from having been completed, but James Bardeen has constructed in successive fractal pictures that attempt to make use of what is known of the dynamics in order to represent the statics. Since very often the geometry of statics is fractal, and the geometry of dynamics is also fractal, fractals do not lose either way.

Q3: I would like to say that I'm completely in agreement with what Q2 just said. I believe that one of the main points is to relate dynamics to chaos and to fractals. In fact, let me give two examples where I think some additional dynamics would be very nice. When we speak about adding some noise, from where is this noise coming? And when we speak about boundary conditions, from where are the boundary conditions coming? Essentially, boundary conditions are an empirical concept. In hydrodynamics or microscopic physics, you can speak about boundaries. If you speak about dynamics, there are no boundaries. Boundaries are part of the dynamical problem. Therefore, in a sense, I think that your presentation, which was very beautiful, of course, is more a kind of phenomenology which has to be, I would say, made a little deeper by making some relation with dynamical concepts.

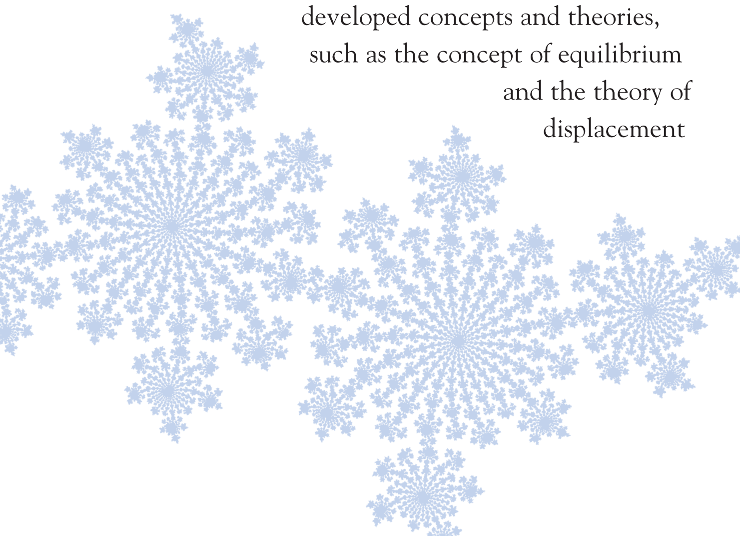
Mandelbrot: Two of the figures illustrated a fractal aggregate. As it grows, its boundary is continually changed by the dynamics of the generating process. Thus, I agree with what you say. This dynamics consists in little particles aggregating together, but eventually leads to an extraordinary structure. The open mystery is why this structure is fractal.

Chairman: In the past, large mathematical models were used to centralize decisions – for example, in economics – for traditional models have not worked. What does the new science bring to prediction, control and, ultimately, to social responsibility?

Mandelbrot: Your question is complicated. I prefer not to answer the last part.

But I have been greatly interested in economics. In view of your comments, I must emphasize that existing economic thought strikingly fail to predict anything about those aspects of the economy on which tests are possible, because data are available in large quantity. For example, many people attempt to explain or predict the stock market, but they all fail. My approach to finance in the early 1960s was very different. It was phenomenological, absolutely, deliberately, and even arrogantly. My goal was to generate wiggles that people active in the stock market would not be able to distinguish from the wiggles they see in newspapers. This goal was both modest and demanding; I succeeded with the help of a very simple, purely random process. Economists challenged me to explain my statistical statics from their dynamics. Disappointingly, their dynamic is not up to the task.

Economics and other more complicated areas borrow a great deal from physics. What they borrow is mostly made of fully developed concepts and theories, such as the concept of equilibrium and the theory of displacement of equilibrium in perfect gases. Next, they try to develop these concepts and themes in rigorous fashion in an economics context. Much less effort is devoted to testing whether economic phenomena really fall into the domain in which those standard physical arguments can conceivably apply.



For example, take continuity. Everyone in economics seemed to assume that prices were a continuous function of time. To the contrary, all the evidence shows that one comes much closer to reality by assuming prices to be a discontinuous function of time. Incidentally, this discontinuity is not that of quantum physics.

To summarize: I was active in economics both in the early 1960s and again more recently. The reason why my effort in this area has been arrogantly phenomenological is because the more ambitious dynamical study of these things has been an abject failure.

Q4: I was interested to see that that question was put in the past tense about complicated economic models. It continues to be true that a fantastic amount of money and effort is put into enormously complex, many variable, mostly linear, economic forecasting models. You can read the predictions from these every year in *The Wall Street Journal*. Models that attempt to link tens of thousands of variables and relationships – home mortgage interest rates, the ratio of the dollar and the yen, the demand for Sierpinski gaskets – anything you can imagine is built into these models, and the results are often announced to two or three digits of precision. And then, of course, next year, they have to be artificially amended with tens of thousands of ad hoc changes. I think we're only beginning to see an appreciation by some economists of some of the work you've already started to describe, and that you'll hear described as this conference goes on. An appreciation of what can be done with a greater recognition of the essential non-linearity of enormous complex systems like economics.

Chairman: I have one more question from the audience: When doing mathematical research, do you discover or invent?

Mandelbrot: I certainly feel that I discover. The assertion that eventually became the four-colour theorem was discovered long ago … by an amateur. It was not some new thing to be invented, but an existing fact to be discovered. It was there.

The same was true when I sat in front of a terminal, next to this extraordinarily gifted young assistant, to investigate the set that became known as the Mandelbrot Set. It was never our feeling that we were inventing anything. This thing was there. My whole thrust was to discover more about its complication. Its complication was the key to the dynamics of quadratic iteration, which is a dynamical system with particularly simple equations. We tried to discover the so-called static geometry of one set, in order to understand the dynamics of another set.

Let me also mention the work on multifractals that I did in the 1960s and published in 1974. In this instance, the process of discovery occurred on two levels. First of all, I discovered new facts about random singular measures. The key was a mathematical theorem that I had learned as a young man, but had always felt would never be used in physics. Hence, it is the study of multifractals that made me discover the real meaning of that theorem. Until then, its statement was so abstract that I could not see it and appreciate what it had always meant.

Proofs are very often a very different matter. Some are so contrived that they definitely look and feel invented, but the best proofs also have both the look and the feel of discovery.

Further reading

The Fractal Geometry of Nature by B.B. Mandelbrot (W.H. Freeman, 1982) was the first comprehensive book on the subject, and remains a basic reference book. Innumerable other books have appeared since. An up-to date list is found on the website www.math.yale.edu/mandelbrot

The basic how-to book is *The Science of Fractal Images*, eds. H.-O. Peitgen and D. Saupe (Springer, 1988).

The best-known book on iteration is, deservedly, *The Beauty of Fractals* by H.-O. Peitgen and P.H. Richter (Springer, 1986).

For other aspects of the mathematics, see *Fractals: Mathematical Foundations and Applications* by K.J. Falconer (Wiley, 1990) and *Fractal Geometry and its Applications: a Jubilee of B. Mandelbrot* ed. M. Lapidus (2004)

On the concrete uses of fractals, three references are convenient, because they are special volumes of widely available periodicals:
1: *Proceedings of the Royal Society of London*, Volume A423 (8 May 1989), which was also reprinted as *Fractals in the Natural Sciences*, ed. M. Fleischmann *et al.* (Princeton University Press, 1990).
2: *Physica D*, Volume 38, which was also reprinted as *Fractals in Physics, Essays in Honor of B.B. Mandelbrot on his 65th birthday*, eds. A. Aharony and J. Feder (North Holland, 1989).
3: *Fractals* Volume 3 (September 1995), reprinted as *Fractal Geometry and Analysis: The Mandelbrot Festschrift, Curação, 1995* eds. C.J.G. Evertsz, H.-O. Peitgen & R.F. Voss.

On the physics, a standard textbook is *Fractals* by J. Feder (Plenum, 1988).