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Multifractal Measures, Especially for the Geophysicist*

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Abstract—This text is addressed to both the beginner and the seasoned professional, geology being used as the main but not the sole illustration. The goal is to present an alternative approach to multifractals, extending and streamlining the original approach in MANDELBROT (1974). The generalization from fractal *sets* to multifractal *measures* involves the passage from geometric objects that are characterized primarily by one number, namely a fractal dimension, to geometric objects that are characterized primarily by a function. The best is to choose the function $\rho(\alpha)$, which is a limit probability distribution that has been plotted suitably, on double logarithmic scales. The quantity α is called Hölder exponent. In terms of the alternative function $f(\alpha)$ used in the approach of Frisch-Parisi and of Halsey *et al.*, one has $\rho(\alpha) = f(\alpha) - E$ for measures supported by the Euclidean space of dimension E. When $f(\alpha) \geq 0$, $f(\alpha)$ is a fractal dimension. However, one may have $f(\alpha) < 0$, in which case α is called "latent." One may even have $\alpha < 0$, in which case α is called "virtual." These anomalies' implications are explored, and experiments are suggested. Of central concern in this paper is the study of low-dimensional cuts through high-dimensional multifractals. This introduces a quantity D_q, which is shown for $q > 1$ to be a critical dimension for the cuts. An "enhanced multifractal diagram" is drawn, including $f(\alpha)$, a function called $\tau(q)$ and D_q.

Key words: Fractal, multifractal, measure, Hölder, limit theorem.

1. Introduction and Motivation. Reasons Why Multifractals are Indispensable in Geophysics and in Other Sciences

The topic of multifractals is bound to become of increasing importance to geophysics, in particular if the present volume becomes influential.

In one phrase, the generalization from fractal sets to multifractal measures involves the passage from geometric objects characterized primarily by one number, to geometric objects characterized primarily by a function. This function can be a probability distribution that has been renormalized and plotted suitably.

In a different single phrase, the generalization for fractal sets to multifractal measures involves the passage from a finite number of fractal dimensions to an

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* Note: This text incorporates *and supersedes* MANDELBROT (1988). A more detailed treatment, in preparation, will incorporate MANDELBROT (1989).

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infinite number of "dimensions." Moreover (and this is a special point of this paper), these "dimensions" can be negative. We shall have gone far from the integer-valued dimensions of Euclid.

1.1. The Example of Copper

Correspondent to the simplest fractals, the basic idea is self-similarity, either exact or approximate. The closely related notions of self-similar fractal or self-similar multifractal can be phrased in many ways, but the geophysicist might best understand them in the context of the distribution of a rare mineral, such as copper. We first consider high-grade copper, then consider gradually lower grades.

High-grade copper is of course distributed nonuniformly: it concentrates in very few regions of the world. If one examines one such region in detail, however, copper continues to be found to be nonuniform: it concentrates in few sub-regions. And so on. It is reasonable, therefore, to suppose that the relative distribution of high-grade copper is the same (in the statistical sense) within each copper-bearing region, whether it is small or large. This being granted, take a large region, and cover it by a grid of equal "cells." As the cell sizes are made smaller, the total area of the cells that contain high-grade copper is found to shrink.

Mathematics has long known a construction that follows this process, and fractal geometry has "tamed" this construction to make it a model of nature. For example see MANDELBROT (1982), *The Fractal Geometry of Nature (FGN)*. In the language of fractal geometry, high-grade copper is usefully viewed as "concentrated on," or "supported by," a self-similar fractal set of low fractal dimension.

Next, examine lower grade copper. The fact that it is more widespread in nature is expressed by its being supported by a fractal set of higher fractal dimension.

Overall, in order to give a full representation of the distribution of copper, it is seen that fractals are necessary and that no single fractal set is sufficient. A simple description consists of giving the fractal dimensions corresponding to each of a sequence of grades, as defined by thresholds varying from 0 to a very high value that is rarely exceeded.

The overall idea of the preceding paragraph has been combined with the generalization of the notion of self-similarity from sets to measures, and has thereby led to the notion of self-similar multifractal *measure*. To say that a multifractal *is a measure* and *not a set* is a very important distinction. It will be explained in Section 3.1. Our work on multifractals was initially concerned with the intermittency of turbulence, and was mostly carried out in the period 1968 to 1976, but it had started about 1962. My book *FGN* surveys multifractals on pp. 375–376, but this survey is overly sketchy and is now obsolete.

1.2. An Interesting Old Quote

The simplest of all multifractals, which is nonrandom, is called binomial and is discussed in Section 5. This construction that has long been known to mathematicians, and has been tamed by fractal geometry, to make it a model of Nature. It happens that the basic circumstances that call for the binomial multifractal measure are very intuitive, and have nearly been rediscovered in the earth sciences context described in Section 1.1. Indeed, the geologist DE WIJS (1951) (quoted in *FGN*, p. 376) has described them as follows:

"Consider a [body of ore] with a tonnage W and an average grade M. With an imaginary cut we slash this body into two halves of equal tonnage $\frac{1}{2}W$, differing in average grade. Accepting for the grade of the richer half $(1+d)M$, the grade of the poorer half has to be $(1-d)M$ to satisfy the condition that the two halves together average again M ... A second imaginary cut divides the body into four parts of equal tonnage $\frac{1}{4}W$, averaging $(1+d)^2M$, $(1+d)(1-d)M$, $(1+d)(1-d)M$, and $(1-d)^2M$. A third cut produces $2^3 = 8$ blocks, namely 1 block with an average grade of $(1+d)^3M$, 3 blocks of $(1+d)^2(1-d)M$, 3 blocks of $(1+d)(1-d)^2M$, and one block of $(1-d)^3M$. One can visualize the continued division into progressively smaller blocks ... The coefficient d as a measure of variability adequately replaces the collective intangibles [dear to those who feel that ore estimation is an art rather than a science], and statistical deductions based upon this measure can abolish the maze of empirical and intuitive techniques."

Of course, de Wijs did not even begin to explore the geometric aspects of his first sketch of a model, and neither he nor notable followers (including G. Matheron) had an inkling of fractals or of multifractals, e.g., of the basic notion of fractal dimension. However, assume that the ore density is independent of grade, making tonnage equivalent to volume, and allow the (reinterpreted) scheme of de Wijs to continue *ad infinitum*. We shall see that this leads to the conclusion that the ore "curdles" into a binomial multifractal.

1.3. Relative Intermittency in a Context Broader than that of Metals

To broaden the scope of multifractals, let us quote from the subsection on *Relative Intermittency* of my book *FGN*, p. 375 ss.

"The phenomena to which [multi] fractals are addressed are scattered throughout this Essay, in the sense that many of my case studies of natural fractals negate some unquestionable knowledge about Nature.

"We forget in Chapter 8 that the noise that causes fractal errors weakens between errors but does not desist.

"We neglect in Chapter 9 our knowledge of the existence of interstellar matter. Its distribution is doubtless *at least* as irregular as that of the stars. In fact, the notion that it is impossible to define a density is stronger and more widely accepted for interstellar than stellar matter. To quote deVaucouleurs, 'it seems difficult to believe that, whereas visible matter is conspicuously clumpy and clustered on all scales, the invisible intergalactic gas is uniform and homogeneous . . . [its] distribution must be closely related to . . . the distribution of galaxies . . . .'

"And in Chapter 10 the pastry-like sheets of turbulent dissipation are an obviously oversimplified view of reality.

"The end of Chapter 9 mentions very briefly the fractal view of the distribution of minerals. Here, the use of closed fractals implies that, between the regions where copper can be mined, the concentration of copper vanishes. In fact, it is very small in most places, but cannot be assumed to vanish everywhere.

"In each case, [portions of space] of less immediate interest were artificially emptied to make it possible to use *closed* fractal sets, but eventually these areas must be filled. This can be done using a fresh hybrid [namely, a] mass distribution in the cosmos such that no portion of space is empty, but, [given two] small thresholds θ and λ, a proportion of mass at least $1 - \lambda$ is concentrated on a portion of space of relative volume at most θ."

1.4. A Feature of Most Direct Importance in Many Sciences: Many Measures are not Observable Directly, only Through "Cuts"

The exploration of the earth cannot be carried out fully in three dimensions. Very often it must follow a straight bore-hole to obtain a straight 1-dimensional cross-cut through a real system that is intrinsically 3-dimensional. Often flat cuts are all that is available for inspection. The same constraint is encountered when turbulence in 3-dimensional space is explored via 1- or perhaps 2-dimensional cuts. Typically, the positions of these cuts bear no relation to the overall turbulence, and can therefore be thought of as having been chosen at random.

Consider also the context of strange attractors. Their full natural space has a very high dimensionality. But they are typically examined via a "Poincaré section" by a plane. The position of the plane, again, often bears no relation to the full attractor, and can be viewed as having been chosen at random. As we shall see by examining typical cases, the measure observed along a random cut has properties that are without counterpart in the measures studied in their natural space, and vice versa. This raises the issue of what can and what cannot be inferred from a cut to the whole measure. This issue is extraordinarily important and has motivated our early work of 1968–1976, especially MANDELBROT (1974).

2. Two Alternative Summaries

The present text begins with introductory material, continues with the binomial measure (Section 5) and then proceeds to step by step generalizations. One must wait until Section 8 to initiate a discussion of the cuts. The result is longer than we would have preferred.