Proceedings of Symposia in
PURE MATHEMATICS

Volume 72, Part 1

Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot

Analysis, Number Theory, and Dynamical Systems

Michel L. Lapidus (Managing Editor)
Machiel van Frankenhuijsen
Editors

American Mathematical Society
Providence, Rhode Island

Proceedings of a special session held in January 2002 during the Annual Meeting of the American Mathematical Society in San Diego, California entitled *Fractal Geometry and Applications*: *A Jubilee of Benoît Mandelbrot.* Some of the contributions to this volume are by speakers from a related special session on *Fractal Geometry, Number Theory, and Dynamical Systems* held during the first Joint Meeting of the Société Mathématique de France and the American Mathematical Society at the École Normale Supérieure de Lyon in July 2001.

2000 *Mathematics Subject Classification.* Primary 28A12, 28A80, 11K60, 11M41, 37A45, 37F45, 60G57, 60J45, 68U05, 82C41.

Library of Congress Cataloging-in-Publication Data

Fractal geometry and applications : a jubilee of Benoît Mandelbrot / Michel L. Lapidus (managing editor), Machiel van Frankenhuijsen, editors.

p. cm. — (Proceedings of symposia in pure mathematics ; v. 72)

Includes bibliiographical references.

Contents: pt. 1. Analysis, number theory, and dynamical systems – pt. 2. Multifractals, probability and statistical mechanics, applications.

ISBN 0-8218-3292-1 (set : acid-free paper) – ISBN 0-8218-3637-4 (pt. 1 : acid-free paper) – ISBN 0-8218-3638-2 (pt. 2 : acid-free paper)

1. Measure theory–Congresses. 2. Fractals–Congresses. 3. Ergodic theory–Congresses. I. Mandelbrot, Benoit B. II. Lapidus, Michel L. (Michel Laurent), 1956– III. van Frankenhuysen, Machiel, 1967– IV. Series.

QA325.F73 2004
515′.42–dc22 2004057084

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 09 08 07 06 05 04

Contents

Part 1. Analysis, Number Theory, and Dynamical Systems

ANALYSIS

NUMBER THEORY

Contents

Part 2. Multifractals, Probability and Statistical Mechanics, Applications

Preface

The goal of this volume is to present to interested mathematicians and other scientists a cross-section of recent research in the field of fractal geometry and its applications, either within mathematics or to other sciences. The volume itself arose in part as the proceedings of a special session held in January 2002 during the Annual Meeting of the American Mathematical Society in San Diego, California, and entitled *Fractal Geometry and Applications*: *A Jubilee of Benoît Mandelbrot*. [The award (to MLL) of a grant by the Office for Naval Research (ONR–N00014-02-1-0168) to partially support this conference and its aftermath is hereby gratefully acknowledged.] The purpose of that conference was to bring together leading researchers working in this field as well as to honor Benoît Mandelbrot on the occasion of the jubilee of his thesis. A more detailed description of the goals and contents of this two-part book is provided in the long general introduction to this volume written by one of us (MLL) and placed just after this preface (in Part 1).

We hope that the mixture of tutorial articles, research expository papers and research original articles found in the two parts of this volume will be useful to the experts and the non-experts alike (including graduate students and postdocs). It should, in particular, demonstrate the vitality and diversity of the field of fractal geometry (taken in a broad sense) and hopefully motivate newcomers to further investigate some of the many open problems and potential research directions proposed throughout the volume.

June 2004
Michel L. Lapidus
Machiel van Frankenhuijsen

List of Tributes

Proceedings of Symposia in Pure Mathematics
Volume **72.1**, 2004

Fractal Geometry and Applications–An Introduction to this Volume

Michel L. Lapidus

1. Introduction

Fractal geometry is a subject that has enjoyed substantial growth over the past decade or so, and has established connections with many areas of mathematics (including harmonic analysis, potential theory, partial differential equations, probability theory, operator algebras, number theory and dynamical systems).

It is also intrinsically a cross-disciplinary subject, with motivations from and applications to physics, engineering, computer science (including computer graphics), chemistry, biology, geology, economics, and even some artistic fields, like painting and music. Of course, one should hasten to say that as is often the case in science, not all of these applications are successful or universally accepted by the practitioners of those fields. Nevertheless, many questions concerning complex natural shapes or phenomena that previously seemed out of reach or simply unreasonable can now be formulated in a mathematical and geometric language and in some cases, can be partly answered.

Similarly, within mathematics proper, there has been and continues to be some significant and even understandable resistance to the use of the term 'fractal geometry', perhaps because of its popularity among a certain public and some of the practitioners of the above 'applied' fields. One additional reason for this is the unfortunate fact that the discussions of fractal geometry found in the press or in popular books quite often fail to place the subject in a historical context,

2000 *Mathematics Subject Classification.* Primary 26A27, 26A30, 28A80, 60J65, 60K40, 28-03, 01-06; Secondary 11-06, 28A75, 28A78, 37-06, 37F45, 42-06, 42A32, 54-06, 82-06.

Key words and phrases. Fractal geometry, mathematical (and physical) sources, fractality, highly non-differentiable functions, Cantor sets and functions, Brownian motion, curves without tangent, random and deterministic fractals, strict and statistical self-similarity, fractal dimensions and measures, wavelets, random curves and surfaces, harmonic analysis, number theory, dynamical systems, multifractals, probability and statistical physics, applications.

This work was partially supported by the US National Science Foundation under the research grant DMS-0070497. Support by the Centre Emile Borel of the Institut Henri Poincaré (IHP) in Paris and by the Institut des Hautes Études Scientifiques (IHÉS) in Bures-sur-Yvette, France, during the preparation of this article is also gratefully acknowledged.

and to show how it connects with and springs naturally from many older and now well-established fields of mathematics.

There is no doubt, however, that the subject of fractal geometry—taken in the broad sense adopted in this volume—is steadily maturing and deepening, while at the same time extending its scope into the mathematical mainstream. In fact, the adjective or name 'fractal' has become ubiquitous in many parts of the mathematical or scientific literature. Although it often refers to vastly different situations, it is typically used to evoke the notion of complexity, roughness, raggedness, scaling, self-similarity or more generally, self-alikeness (strict or approximate, random or deterministic). We do not intend here to enter into the long-term debate regarding the precise mathematical meaning of the word 'fractal'. Instead, we prefer to refer the interested reader to the introduction of Falconer's book from which the following quote [**Fa1**, p. xx], is excerpted (also see, for example, [**La-vF**, § 10.2] and the relevant references therein for a related discussion and complementary viewpoints):

> *My personal feeling is that the definition of a 'fractal' should be regarded in the same way as the biologist regards the definition of 'life'. There is no hard and fast definition, but just a list of properties characteristic of a living thing, such as the ability to reproduce or to move or to exist to some extent independently of the environment. Most living things have most of the characteristics on the list, though there are living objects that are exceptions to each of them.*

2. Some Mathematical Sources of Fractal Geometry

As is well-known, several important mathematical aspects of fractal geometry have their origins in various analytical and geometric constructions of nineteenth or early to mid-twentieth century mathematics. The latter were aimed in part at finding the 'natural boundaries' and/or hypotheses of still evolving concepts or of classic or recently formulated 'theorems' and conjectures involving, for instance, the notions of continuity, differentiability, tangent line (or plane) to a curve (or surface), and later on, Lebesgue measure and its generalizations (including the Hausdorff–Besicovitch measure) as well as different types of 'dimensions' in topological or metric spaces.

Fourier (or more generally, harmonic) analysis—developed in part by the engineer, mathematician and physicist (as well as politician) Joseph Fourier ([**Fo**], 1812 & 1822) in order to study the temperature distribution in various media via the solutions of the heat (or diffusion) equation—has played and continues to play a prominent role in this respect. It motivated, in particular, mathematicians like Bernhard Riemann ([**Ri**], in about 1861), Karl Weierstrass ([**We**], in 1872) and others to introduce (via the use of explicit trigonometric series) and study their famous examples of continuous but highly non-differentiable (or even nowhere differentiable) functions.[1] Before those works, a widespread belief was that every

[1] It seems that even before 1830, Bernard Bolzano [**Bol**] had given an example with analogous properties. Moreover, Godfroy Hardy ([**Har**], 1916) has pursued, in particular, the study of Riemann's example; see, e.g., Stéphane Jaffard's article on multifractals in Part 2 of this volume for a discussion of (and references to) more recent results along those lines. Finally, the rigorous study of trigonometric series really began in the late 1820's with the work of Johann P. G. Lejeune

'function' given by a 'closed analytic formula' should be differentiable at 'most' points.

Similarly, the groundbreaking work of Georg Cantor (in the 1880's) on the theory of sets, cardinals and ordinals, led as a natural corollary to a complete rethinking of the notion of 'sameness' and 'complexity' in that context, along with a significant broadening of the type of sets and functions one might have to deal with in mathematics and in analysis, in particular. (Again, it is noteworthy that one of Cantor's motivations for developing his theory was the study of concrete problems regarding 'sets of uniqueness' and related issues connected to trigonometric series; see, e.g., [**Da**].) The now ubiquitous Cantor set and associated Cantor function (the graph of which was since then coined colorfully by Mandelbrot [**Ma5, Ma7**] the 'Devil's staircase' (or *escalier du diable* in French) were born out of that work; see ([**Can**], 1884) where they were both introduced.[2]

In modern terminology, the Cantor set provided an example of a perfect subset of the interval [0,1] that has the same cardinality as [0,1] itself (or as the real line) but has zero (one-dimensional) Lebesgue measure (or 'length'), while the Cantor function was an example of a continuous, non-decreasing function on [0,1] that grows from 0 to 1 and is constant on every open interval in the complement of the Cantor set. From a pedagogical point of view, the latter yields a simple example of a continuous function that is differentiable and has zero derivative almost everywhere on [0,1] but is nevertheless non-constant on the entire interval [0,1]: a very salutary warning that a key corollary to the Fundamental Theorem of Calculus could not be too naively extended. This was later on profitably taken into account by Henri Lebesgue [**Leb1**]–[**Leb3**] (see also [**Haw**]) in obtaining his counterpart of that theorem within the context of his theory of integration.

All the above examples have been considered as being literally 'monstrous' by many leading mathematicians contemporary to Riemann, Weierstrass, or Cantor.[3] They have therefore been the source of considerable controversy and debate within the mathematical community and beyond, in some ways akin to (although perhaps less dramatic than) the culture shock provided by Gödel's discoveries in logics and the foundations of mathematics in the 1930's. On the other hand, they have stimulated the imagination and creativity of other mathematicians and scientists who recognized intuitively that a much richer world was perhaps hidden beyond the familiar Euclidean–Newtonian and deterministic universe, with possible relevance to physical reality. Among these researchers, it may be fitting to single out the physicist Jean Perrin whose pioneering experimental work and observations on

---

Dirichlet (of whom Riemann was a student) who not only gave the modern definition of a 'function' (in about 1837) but also studied the convergence of Fourier series in a very precise manner; see, e.g., [**Di1, Di2**].

[2]In fact, the 'Cantor set' itself had been discovered independently about ten years earlier by another (less well-known) mathematician (H. Smith, in 1875 [**Smi**, Chapter XXV]). (See, e.g., the relevant historical notes appended to [**Ma7**], pp. 408–409.) We will usually omit further mention of such facts in the sequel because we would like to focus here on some of the historical roots of fractal geometry rather than on the individual contributors to it.

[3]An often-quoted text in this regard is the following excerpt from an 1893 letter from Charles Hermite to Thomas Stieltjes [**HerSt**, p. 318]: *"Je me détourne avec effroi et horreur de cette plaie lamentable des fonctions continues qui sont sans dérivée..."*; ["I turn away with fright and horror from this awful plague brought upon by continuous functions which do not have a derivative..."]; see also the interesting related discussion in [**GrKa**].

Brownian motion inspired the following comments in his classic book *Les Atomes* ([**Per**, p. 27 & p. 25] 1913; quoted and translated in [**JoLa**, p. 24 & p. 62]):

> *We will stay within the realm of experimental reality if, putting our eye on a microscope, we observe the Brownian motion which agitates every small particle suspended in a fluid. In order to obtain a tangent to its trajectory, we should find a limit, at least approximately, to the direction of the line which joins the position of this particle at two successive instants very close to each other. But, as long as one can perform the experiment, this direction varies wildly when we let the duration that separates these two instants decrease more and more. So what is suggested by this study to the observer without prejudice, is again the function without derivative, and not at all the curve with a tangent.* (...)
>
> *Mathematicians, however, have well understood the lack of rigor of such considerations, said to be geometric, and how, for example, it is childish to want to prove, by simply drawing a curve, that every continuous function admits a derivative. If the differentiable functions are the simplest, the easiest to deal with, they are nevertheless the exception; or if one prefers a geometric language, the curves which do not have a tangent are the rule, and the nicely regular curves, such as the circle, are quite interesting, but very particular cases.*

In the 1920's, Norbert Wiener [**Wi1**]—only a few years after the publication of Lebesgue's theory of integration and of Volterra's functional analytic work on 'functions of line' (or 'functionals')—provided a remarkable mathematical model for Brownian motion, the physical theory of which had been developed by Einstein [**Ei**] in 1905 and independently (but unknown to Wiener) by Smoluchowski [**Smo**] at about the same time.[4] The resulting Wiener probability measure and the associated Wiener process (often referred to as 'Brownian motion' by contemporary probabilists, as is the case in Part 2 of this volume)—discovered almost ten years before Kolmogorov's axiomatization of probability theory [**Kol**]—is now a primary example of a stochastic Markov process with continuous paths. Almost surely, Brownian paths (i.e., the sample paths of the Wiener process) are 'fractal' (in several senses), continuous but nowhere differentiable, 'statistically self-similar', and have an extraordinarily complicated and wonderfully rich fine geometric structure.[5] Along with its many (discrete or continuous) generalizations, Brownian motion provides an endless reservoir of random geometric fractal objects (random curves, surfaces, Cantor sets, etc.). Nowadays, random fractals—including statistically self-similar (or rather, self-affine) fractals—are the object of intensive investigation and play a key role as mathematical models for physical phenomena approached

[4]Interestingly, even earlier, in his 1900 Ph.D. dissertation on pricing and on speculation in financial markets [**Ba**] (for which Henri Poincaré was an unimpressed referee), Louis Bachelier had made important contributions to the mathematical theory of Brownian motion. His pioneering work was not recognized for many years, however, but is now widely quoted by mathematical economists and other scientists. (See, e.g., the relevant historical notes to [**Ma7**], pp. 392–395.)

[5]We refer the interested reader, e.g., to Chapters 2–5 of [**JoLa**] (and to the relevant references therein) for a brief introduction to the fascinating history of the study of Brownian motion as well as for a mathematical discussion of the Wiener process and of some of its key properties, from an analyst's viewpoint; see also, e.g., [**Ne**], [**BenA**] and [**Lev1**].

via fractal geometry. Many illustrations of this statement are provided in Part 2 of this volume, both from the mathematical and physical point of view.[6] We point out that a simple and enlightening early example of the use of 'statistical self-similarity' was given by Benoît Mandelbrot in his famous 1967 article [**Ma2**], entitled *How long is the coast of Britain? Statistical self-similarity and fractional dimension.*[7]

Our brief discussion would not even begin to be complete without mentioning the examples provided by Giuseppe Peano (the so-called 'Peano plane-filling curve', see [**Pe**], 1890), the baron Helge von Koch (the Koch 'snowflake curve',[8] see [**vK1**]–[**vK2**], 1904 & 1906), Waclaw Sierpinski (namely, the Sierpinski 'gasket' and 'carpet', see [**Si**], 1916), Karl Menger (the Menger 'sponge', a three-dimensional analogue of the Sierpinski carpet, see [**Me**], 1926), David Hilbert, Tagaki ([**Ta**], 1903) and van der Waerden (1930), Paul Lévy (Lévy's 'dragon curve', see [**Lev2**], 1938), and Georges de Rham ([**dR**], 1957), among many others. These examples of (deterministic) curves or surfaces without tangent lines or planes (almost everywhere), often presenting self-similarity properties, have also been considered as 'pathological monsters' [**Ma5, Ma7**] or simply ignored by many mathematicians at the time but now belong to the toolkit of every fractal geometer. The interested reader may find pictures and more information about these and related examples in [**Ma5, Ma7, Fa2, Fa3, PeiJS**]. It is noteworthy that Sierpinski's main purpose in studying the Sierpinski carpet (or square)[9] in [**Si**] was to show its 'universality' among all planar curves; roughly speaking, this means that every planar Jordan curve can be embedded homeomorphically into the Sierpinski carpet (a property extended to higher dimensions by Menger in [**Me**] but that is not shared, for instance, by the Sierpinski gasket (or triangle)); see, e.g., [**PeiJS**], [**La6**] and the relevant references therein (including [**Pea**]) for further discussion of this point.

In closing this section, let us mention two other important mathematical sources of contemporary fractal geometry:

First, the study of various metric notions of dimensions and of the associated measures (when applicable). Among those, we point out the classic Hausdorff dimension and measure, introduced by Felix Hausdorff in ([**Hau**], 1918) and then

---

[6]According to a very nice phrase of Bernard Sapoval (oral communication), *"La géométrie fractale constitue une cristallographie du hasard"* ["Fractal geometry provides a crystallography of chance"].

[7]Roughly speaking, a (deterministic) geometric object (embedded in Euclidean space, say) is said to be 'self-similar' if it can be decomposed into a finite number of pieces, each of which is 'similar' to the whole (or is a 'scaled copy', up to translation or rotation, of the original set). More generally, one can talk about sets that are 'self-affine' or 'self-alike' (in some suitable sense), as well as of their random counterpart. (See, e.g., [**Mo, Hu**], [**Ma7**] and [**Fa1, Fa2**].)

We note that as Mandelbrot often points out in his public lectures, the (albeit unformalized) notion of 'self-similarity' probably goes back very far in mankind's history, witness the motives and patterns occurring in certain monuments, potteries, mosaics or paintings, for example. Moreover, much more recently and if it is interpreted in the broader sense of 'scaling law', it has played a crucial bridge between fractal geometry and various physical applications, for instance, via Wilson's renormalization theory [**Wil**] and the study of critical phenomena in quantum field theory and statistical physics; see, e.g., [**Col**], [**AhFe**], [**Sc**] and [**PeiR**]. Aspects of the latter theme are illustrated in Part 1 and Part 2 of this book.

[8]the study of which was pursued, in particular, by Ernest Cesàro in [**Ce**] and by Paul Lévy in his remarkable paper [**Lev2**].

[9]itself a two-dimensional analogue of the ternary Cantor set and apparently introduced earlier by the Polish mathematician Stefan Mazurkiewicz in 1913.

studied or applied extensively during the following decades by Abraham Besicovitch and his collaborators. (See, e.g., [**Bes1**]–[**Bes4**], [**BesUr**], and [**BesTay**]; also see [**Ro**] and [**Fa1, Fa2**] for a modern exposition.)[10] Further, we mention another notion of 'fractal dimension', namely, the Minkowski–Bouligand dimension (also called 'box dimension' or 'ϵ-entropy', among many other names), essentially introduced in its modern form by Georges Bouligand in ([**Bou**], 1928) and about which there has been until recently some confusion in the applications of fractal geometry. (See, e.g., [**Ma7, Fa2, Fa3, Fed, Tr1, Tr2, La1, La-vF, Mat**], for a discussion of these different kinds of fractional dimensions and of extensions thereof.)[11] (Also see, in particular, the 1932 article by Pontrjagin and Schnirelmann [**PonSch**], along with the 1959 paper by Kolmogorov and Tihomirov [**KolTi**].) We note that many of the historical articles mentioned so far in this section can be found (in English translation and with helpful accompanying comments) in the interesting book edited by Gerald Edgar [**Ed**], *Classics on Fractals*.

Second, the study of the iteration of complex mappings (e.g., polynomials or more generally, rational functions). We limit ourselves here to mention the now well-known parallel (and competing) works by Pierre Fatou (e.g., [**Fat1, Fat2**]) and Gaston Julia (e.g., [**Ju1**]), which are remarkable in a number of ways and were written in the early 1920's many years before the advent of the modern computer. An excellent source for the early history of this subject is the book by Daniel Alexander [**Al**], *A History of Complex Dynamics*, subtitled *From Schröder to Fatou and Julia*. (Also see, e.g., the recent book [**Ma11**] for Mandelbrot's own perspective on the subject, which has known a spectacular rebirth after the publication in 1980 of the article [**Ma6**] on what later came to be known as the 'Mandelbrot set'.) Various aspects of the relationship between fractal geometry and dynamics, in particular complex dynamics (including Julia sets, the Mandelbrot set, and their generalizations) are explored in Part 1 of this volume.

3. A Brief Overview of the Goals and the Contents of this Volume

As was alluded to in several places in the previous section, many of the themes that have given rise to and historically nurtured fractal geometry are illustrated and amplified in various ways in this volume, as will be further discussed below. Some other themes have appeared more recently and are also developed.

The goal of this book is to provide a curious and mathematically knowledgeable reader with an introduction to the subject of fractal geometry and to some of its applications, either to other areas of mathematics or to the other sciences, particularly, physics, engineering and computer science. It is hoped that no matter

[10] Hausdorff's work relied on the concept of outer measure then recently introduced by Constantin Carathéodory in ([**Car**], 1914) and itself viewed as a natural generalization of Lebesgue's notion of 'set function' or measure. (See, e.g., [**Co**] and [**Ro**].) Moreover, it is noteworthy that Besicovitch's work was itself motivated in part by problems coming from number theory and from the theory of Diophantine approximation, in particular (see esp. [**Bes2**]–[**Bes4**]). This is one of the themes of Part 1 of the present volume in the chapter on 'Number Theory'; see, especially, the papers by Dodson and Kristensen as well as by Lapidus and van Frankenhuijsen in that chapter, for different perspectives (and goals).

[11] In the context of the study of 'fractal drums' [**Ber**] (or rather, 'drums with fractal boundary'), in particular, it is important to distinguish between Hausdorff and Minkowski dimension, the latter notion being better suited to this problem; see, e.g., [**BrCa**], [**La1**]–[**La5**], [**LaPo1**]–[**LaPo2**], [**LaMai**], [**HeLa**], [**La-vF**], [**HaLa**], [**LaPea**] and the relevant references therein.

what his or her initial level of expertise in this general area, the scientifically literate reader will be able to find texts that are interesting and informative, be they tutorial articles, research expository articles, or entirely original research articles. Great care has been put in the selection, the refereeing and the editing of the papers, written by a diverse group of internationally renowned experts, well-established researchers as well as promising junior researchers.[12] Most of the contributors were invited speakers at the special session bearing the title of this volume[13] and held during the Annual Meeting of the American Mathematical Society in San Diego (January 2002), while others spoke at a related special session on *Fractal Geometry, Number Theory, and Dynamical Systems* held during the first Joint Meeting of the Société Mathématique de France and the American Mathematical Society at the École Normale Supérieure de Lyon (July 2001).[14]

An important objective of this volume is to provide a representative cross-section of the richness and variety of the contemporary research performed in and around fractal geometry. Needless to say, there is no implied claim of completeness and hence, absolutely no negative inference should be drawn from the absence of a particular topic or area. Also, as is unavoidable in such situations, several important topics are missing because unfortunately, the corresponding invited speakers did not have the time to write-up their contribution or to complete it by the final submission deadline. (Among the applications, it is to be regretted that for example, the relationship between aspects of fractal geometry and biology are not touched upon; see, e.g., [**LoMNW**] for a thorough discussion of this subject.) Rather than using this book as an encyclopedia (which it is definitely not), we would like the reader to enjoy the diversity of the themes encountered in this volume. We hope that the reader would then be encouraged, for instance via one of the expository articles and the associated bibliography, to find out more about a particular topic and possibly, to engage in research or intellectual exploration of his or her own, beyond the limits of our current knowledge. In a literal sense, this is an 'open book' and should be used as such.

Let us now proceed to briefly discuss the contents of the main body of this book. The volume is divided into two parts, each of which is subdivided ino three (unnumbered) chapters. Of course, there is some arbitrariness in the placement of a particular article in the volume. Indeed, there is a significant overlap between the themes of Part 1 and Part 2, and within a given part, the boundary between the various chapters is itself often blurred.

---

[12]In particular, over the last two years, as the managing editor of this volume, once the refereeing process was positively completed for a particular article, the author of the present introduction has spent countless hours reading and commenting on the paper, often leading to many successive versions which the contributors have been kind enough to provide in order to take his remarks into account. I would like to take this opportunity to sincerely thank all the contributors of this volume along with all the many conscientious referees for their patience as well as for their generous and precious cooperation in this time-consuming endeavor. Hopefully, the results of this long process will end up being beneficial to the reader.

[13]Some of them could not come to the meeting but were still able to contribute a paper.

[14]The former special session was organized by the editors of this volume, while the latter was organized by the editors along with Michel Mendès France.

The first part (Part 1) of this volume is subtitled *Analysis, Number Theory, and Dynamical Systems* and presents a broad selection of subjects in those fields of mathematics.

The first chapter of Part 1, titled *Analysis*, begins with a poetic and mathematical study by Michel Mendès France of the fractal patterns made, for example, by ripples of light at the surface of the water. It continues with a study by Marc Frantz of Mandelbrot's elusive notion of 'lacunarity' or 'texture' (in the one-dimensional case), followed by a short article by Frank Morgan on the connections between fractal geometry and geometric measure theory ("friends and foes"). The following article, by Hillel Furstenberg and Yitzhak Katznelson, uses tools from harmonic analysis and ergodic theory to investigate a suitable notion of 'multiplicity' associated with a self-similar (or self-affine) fractal. The next three articles, by Atsushi Kameyama, Alexander Teplyaev, and Christophe Sabot, respectively, explore various aspects of 'analysis on fractals' (see, e.g., [**Bar**], [**Ki**]). The first one is an exposition of results obtained on metrics naturally associated with 'topological self-similar sets' (not necessarily embedded in Euclidean space and coded by symbolic dynamics), while the second is a survey of results connected with 'the' Laplacian and the associated Dirichlet form (or 'energy functional') on the Sierpinski gasket; the goal of the third one (by Sabot) is to present its author's intriguing recent theory of the 'renormalization operator' associated with certain self-similar fractals (and defined via multidimensional complex dynamics), while stressing new connections with well-known constructions from symplectic geometry.[15] Finally, the tutorial article by Boris Solomyak gives a nice introduction to various aspects of the theory of Bernoulli convolutions and certain self-similar measures on the real line. It addresses, in particular, the so-called 'Erdös problem' for such measures and involves the consideration of suitable algebraic numbers playing an important role in aspects of harmonic analysis, like the PV (or Pisot–Vijayvaraghavan) numbers.

Next, the chapter on *Number Theory* begins with a paper by Titus Hilberdink also dedicated to the study of Bernoulli convolutions, but now from the point of view of analytic number theory. It is followed by a paper by Stéphane Jaffard using tools from multifractal geometry and wavelet theory in order to investigate certain Dirichlet series, called 'Davenport expansions'. The informative tutorial article by Maurice Dodson and Simon Kristensen explores some of the connections between 'Hausdorff dimension and Diophantine approximation', beginning with the work of Besicovitch and his collaborators referred to in Section 2 of this introduction. Finally, the research expository article by Michel Lapidus and Machiel van Frankenhuijsen gives an overview of the theory of complex fractal dimensions developed by these authors, with emphasis on the study (via suitable Diophantine approximation techniques) of the 'quasiperiodic patterns' of the complex dimensions of self-similar fractal strings and a discussion of their possible cohomological interpretation.

The third and last chapter of Part 1, entitled *Dynamical Systems*, is devoted to various aspects of the relationship between fractal geometry and real or complex (as well as discrete or continuous) dynamics. In the first article, Byungik

[15] As will be clear to the reader, this last research expository article could also have been placed in the dynamics section of Part 1 or in the probabilistic section of Part 2. Similarly, the research article by Furstenberg and Katznelson, for example, could have been included in the chapter (or 'section') on dynamical systems.

Kahng studies certain fractals associated with 'symplectic piecewise affine elliptic dynamics' and proposes several open problems in this area. In the second paper, Sylvain Crovisier surveys recent results obtained about (almost sure) rotation numbers associated with homeomorphisms of the circle (for example), and the corresponding Cantor-type functions (or 'staircases'). Then, Viviane Baladi investigates the spectrum of the (Ruelle) transfer operator in the higher-dimensional case and explores an associated trace formula. (Examples of real and discrete dynamical systems motivating this work include the iteration of so-called 'interval-exchange maps' coded by 'kneading' sequences.) Furthermore, Valentin Afraimovich, Leticia Ramírez and Edgardo Ugalde study various spectra of generalized dimensions associated to '(nonuniformly) hyperbolic geometric constructions', arising naturally in the study of nonlinear (and chaotic) dynamical systems. (See, e.g., [**Pes**].)

The last two papers in this chapter deal with aspects of complex dynamics, a subject that has played an important role in the emergence of fractal geometry, as was briefly mentioned at the end of Section 2. Both can be viewed as tutorial (or research expository) articles in their field. In the first one, Mark Comerford gives an overview of a relatively new and unfamiliar area of complex dynamics, that of 'random iterations'; this means that one considers the dynamical system associated with the (deterministic) iteration of a suitable sequence of complex polynomials of a complex variable (rather than with the iteration of a single polynomial). The notions of Julia set, Fatou set, and related concepts can be suitably adapted to this more general context and several of their properties are investigated. Finally, in the last article of this chapter, Dierk Schleicher discusses the extension of the classical theory of the Mandelbrot set (associated with the iteration of quadratic polynomials, see, e.g., [**Lei**]) to that of so-called 'Multibrot sets' (corresponding to the iteration of polynomials of higher degree). Sketches of proofs of the main results are provided and useful techniques are reviewed or introduced. Further, aspects of the long-standing problem of the simple connectivity of the Mandelbrot set (and its generalizations) are addressed.

Now we come to Part 2 of the volume, which is subtitled *Multifractals, Probability and Statistical Mechanics, Applications* and is also divided into three chapters.

The first chapter of Part 2 is titled *Multifractals*. The first three articles (Part I—III) of this chapter should be viewed as a whole. They provide an introduction to and further develop the theory of *random multiplicative multifractal measures*. The first two of these articles are co-authored by Julien Barral and Benoît Mandelbrot while the third one (which is devoted to the proof of the main results) is authored by Julien Barral. They should be very useful to someone interested in finding out more about Mandelbrot's own approach to and intuition regarding this important subject, beginning with his original articles [**Ma3, Ma4**]. (Also see the recent book [**Ma9**].) Probability techniques and reasoning are an integral part of this approach, that has been developed by many mathematicians, physicists, engineers and other scientists (see, e.g., the combined bibliography to these articles). For the experts, it is worth noting that these three papers contain many new results and techniques that should play an important role in the future developments of the subject.

Next, the reader will find a long tutorial article by Stéphane Jaffard on *Wavelet techniques in multifractal analysis* that provides a very readable introduction to the relationship between multifractals and a recent form of harmonic analysis—namely, wavelet expansions, which are a useful substitute for Fourier series (or integrals) when scaling considerations are involved. It may be helpful to recall at this point that the notion of a 'multifractal' is an extension of the standard notion of a homogeneous fractal; indeed, the fractal dimension is now allowed to vary locally. In particular, instead of giving to a multifractal a single dimension, one associates to it a (typically continuous) spectrum of dimensions, along with a suitable curve (a physical interpretation of which is provided by the 'thermodynamical formalism' for statistical physics). (Many natural fractals are of this type; see, e.g., [**AhFe, Fe, Sc, Ma9**].) As was alluded to in Section 2, one of the applications discussed in this paper is to the determination of the 'multifractal spectrum' (or from the present point of view, the 'spectrum of singularities') of Riemann's nondifferentiable function given by a trigonometric series. The next paper, by Jacques Lévy Vehel and Stéphane Seuret, is an equally long tutorial article that revisits many of the same themes as the previous article, but now from the somewhat different and complementary point of view of the '2-microlocal formalism'. A variety of interesting examples are provided and a comparison with other formalisms is also included. The last paper in this chapter is a research article by Jacques Peyrière in which is proposed a very general 'vectorial multifractal formalism' applicable to vector-valued (rather than scalar-valued) multifractal measures.

The second chapter of Part 2 of this volume is entitled *Probability and Statistical Mechanics.* Brownian-type motions on Euclidean spaces or on manifolds and their natural analogue in fractal-like spaces—along with their discrete counterpart, random walks—serve as a key paradigm for this chapter. The chapter begins with a research expository article by Ben Hambly and Takashi Kumagai in which the authors develop a parallel between aspects of probabilistic analysis on (self-similar) fractals (see, e.g., [**Bar**]) and random walks on certain (infinite, 'fractal') graphs.[16] One of the main goals of the paper is to obtain suitable estimates for the associated heat kernels. The next paper is a long tutorial article, entitled *Random fractals and Markov processes*, in which Yimin Xiao surveys the relationship between aspects of probability theory (particularly, Markov stochastic processes and their sample paths), potential theory, and dimension theory (including the Hausdorff and packing dimensions and measures), among many other topics. It should be very helpful to the reader interested in the mathematical theory of random fractals. It may also serve as an up-to-date complement to the well-known survey article by Samuel James Taylor [**Tay**], with its account of classical as well as more recent results on hitting probabilities, polar sets, random intersections, multiple points, and Hausdorff measure of the range (of sample paths of Markov processes).

Then, in a research expository article, Gregory Lawler, Oded Schramm and Wendelin Werner survey aspects of their recent fundamental work on the *scaling limit of planar self-avoiding walks* and on conformally invariant fractals. (See, e.g., [**LawSW1, LawSW2**], along with the short expository ICM article [**Law**].) This

[16] 'Brownian motions' or more generally, 'diffusions', on suitable 'finitely ramified' (or p.c.f.) self-similar fractals (such as the Sierpinski gasket or certain 'fractal trees', for example) are defined as appropriately rescaled or renormalized limits of random walks. Analogously, the associated 'Laplacians' (or the infinitesimal generators of the corresponding 'heat semigroups') can be defined as suitably renormalized limits of second order finite-difference operators on the approximating finite graphs. (See, e.g., [**Bar, Ki, KiLa1, La4, St**] and the relevant references therein and in the article under review.)

includes, in particular, a proof of Mandelbrot's 1982 $\frac{4}{3}$-conjecture about the fractal dimension of the effective boundary of 'Brownian highlands' (see, e.g., [**Ma7**] and the cover page along with the accompanying text of [**FrMa**]). More precisely, the latter conjecture states that almost surely, the Hausdorff dimension of the outer (or 'effective')[17] boundary of the infinite connected component of the complement of a planar Brownian path (in Euclidean space) is equal to $\frac{4}{3}$. (The proof of this conjecture is announced in [**LawSW1**] and is given in full in the second paper of reference [**LawSW2**].) A key related problem addressed and resolved in this work is to show the existence and to identify the limiting process of a sequence of self-avoiding random walks (with smaller and smaller steps, that is, in the 'lattice approximation') constrained to remain within a given simply connected planar domain (and subject to suitable boundary conditions). This limiting process is shown to be a so-called 'SLE process' (short-hand for 'stochastic Loewner evolution process'), itself closely related to a planar Brownian motion with drift (via an appropriate stochastic differential equation, called the SLE equation). In fact, the solutions of the SLE equation enable the authors to connect in a very precise manner two apparently quite different problems; namely, the existence and identification of the above limit, on the one hand, and the study of random conformally invariant simple curves, on the other hand.[18] In their survey article, Lawler, Schramm and Werner also discuss a number of open problems and the possible direction of future research in this rapidly expanding area.

The last paper of this chapter (i.e., Chapter 2), written by the theoretical physicist Bertrand Duplantier, is entitled *Conformal fractal geometry & boundary quantum gravity*. It is essentially a self-contained research monograph (close to a hundred and twenty pages long) and provides a fascinating overview of the author's (and his collaborators') life-time work on the study of random conformally invariant (fractal) curves via techniques and methods inspired by electrostatics, conformal field theory, string theory and quantum gravity. Integration over random surfaces (much as in string theory, for example)[19]—rather than over random paths (as in ordinary quantum mechanics)—plays a key role in this approach. So do the associated Feynman diagrams that keep track of the interactions between elementary particles in quantum field theory[20] and help encode combinatorially the 'partition function' or more general 'correlation functions' of the quantum gravity model. A remarkable fact is that much of the physical work described in Duplantier's paper was developed before any strictly mathematical approach was available (or at least able to solve several of the long-standing open problems, as described in the preceding article) and that the predictions made in this '(boundary) quantum gravity approach' go well beyond what is now known to be proved rigorously. In particular, 'duality relations' (inspired in part by the 'dualities' from conformal field theory

[17]Here, physically, the adjective 'effective' refers to the 'visible' part of the equipotential boundary from the point of view of the electrostatic field. Mathematically, it corresponds to the notion of 'harmonic measure' which captures the main features of the electrostatic field near the boundary and is a key tool connecting potential theory and Brownian motion (or diffusions) on planar domains; see, e.g., [**Do**] and [**Pom**].

[18]Originally, the consideration of the (stochastic) 'Loewner evolution equation' was motivated by Loewner's work (carried out in a deterministic context) on univalent functions in complex analysis and conformal geometry.

[19]See, e.g., [**Poly**], [**Wit**], [**Polc**], and [**JoLa**, § 20.2].

[20]See, e.g., the references of the previous footnote along with [**Col, Matt, ItZu**].

and string theory or M-theory) are obtained for various dimensions attached to random conformally invariant fractals and shed new light on aspects of this subject. Moreover, a variety of multifractal spectra (of standard as well as of new types)—attached in a suitable way to the effective boundary of conformally invariant random fractals—are studied and computed rather explicitly.[21] Much more is done or discussed in this paper, which we invite the curious and physically inclined reader to explore on his or her own. We simply mention here that related results are obtained about random walks and aspects of percolation theory. Furthermore, the physical motivations for this work include the study of macromolecules and polymers arising naturally in organic chemistry and biology, as well as the study of critical field theories and second-order phase transitions in quantum statistical mechanics. (It is noteworthy that this 'tutorial research monograph' has been written with an eye towards a mixed mathematical and physical audience.) As was alluded to earlier, many of these results are established at the 'theoretical physicist's level of rigor' and do not yet seem to be within the reach of the present (purely) mathematical techniques. However, they rely on a very original and quite convincing methodology and the predictions made by the physical theory are compatible with our present knowledge. On the other hand, the mathematical work of Lawler, Schramm and Werner surveyed in the previous article involves entirely new concepts and methods (as well as results) that were not previously available in the physical literature. (Examples of such notions are the SLE equation and processes.) We hope that the juxtaposition and friendly 'confrontation' at the end of this chapter of these two important mathematical and physical works on the study of random conformally invariant fractal planar curves (or boundaries) will stimulate further investigations in this very interesting area and will motivate some researchers in the future to explore and help clarify the connections between these two seemingly very different approaches. The resulting cross-fertilization would certainly help fulfill one of the main objectives of this volume.[22]

The third and last chapter of Part 2 (as well as of this volume) consists of three papers dealing with various scientific applications of fractal geometry, to condensed matter physics, chemical engineering, or computer graphics. Clearly, since their emphasis or motivation is of a very different nature, these papers are significantly less mathematical than the rest of the contributions to this volume. The first article of the chapter is written by the solid state and experimental physicists Agnès Desolneux, Bernard Sapoval and Andrea Baldassarri. It deals with several problems related to percolation (in the "etching of random solids") and the associated power laws, both in a 'fractal' or 'non-fractal' regime. The results obtained are compared with physical experiments and numerical simulations. Recent striking mathematical results by Stanislav Smirnov [**Sm**] about (among other topics) critical percolation on triangular lattices are also briefly discussed. This is part of a long series of papers by Bernard Sapoval and his collaborators related to condensed matter physics and

[21] Some of these results have been obtained rigorously in the aforementioned work of Lawler, Schramm, and Werner (or in earlier joint or individual works of some of those authors and their collaborators). For example, the main objective of the first paper in reference [**LawSW2**] is to prove a conjecture of B. Duplantier and K.-H. Kwon [**DuKw**] "about the values of 'intersection exponents' of random walks in a half-plane."

[22] This is neither the place nor the time to provide a detailed comparison between these two approaches. Further, the writer of this introduction does not have the necessary expertise to carry it out.

random or deterministic aspects of fractal geometry. (See, for example, Sapoval's tribute in this volume.)

The next article is written by the chemical and physical engineer Marc-Olivier Coppens. It gives an overview of his original work on the practical use of fractal geometry to design reactors in order to improve the efficiency of chemical reactions. Roughly speaking, a strong catalytic effect is obtained by taking advantage of fractal porous structures to increase the surface of contact between the reactants. It is noteworthy that several patents have been obtained by the author and his collaborators for the actual design of chemical reactors following these prescriptions. Finally, the last article of this chapter (and volume) is entitled *Fractal forgeries of nature* and is written by the computer scientist and artist Kenton Musgrave.[23] This tutorial article gives an interesting overview of the method/program ('MojoWorld') which he has developed over the last ten years (beginning with his 1994 Ph.D. thesis under Benoît Mandelbrot at Yale University and continuing with his collaborators in the computer graphics company he is now leading). Its aim is to use fractal geometry in order to imitate nature. The protypical example is the computer-aided design of planets, mountainous landscapes, or of rough terrains. The ins and outs of 'MojoWorld' are explained, often in very practical terms, and some of the underlying mathematics is discussed. The article is abundantly illustrated by stunningly beautiful computer-generated pictures. Unfortunately, due to financial restrictions, we had to publish them in black and white. It is worth pointing out, however, that the actual color pictures are available on the web site indicated in the short bibliography to the paper and may also be available in the future on an appropriate AMS web site connected with this volume.

This concludes our overview of the volume. We hope that it will help guide the reader through this nearly one thousand page-long and two-part book. We also hope that it gives an idea to the non-expert of the many facets of the general area of fractal geometry and of some of its applications.

4. The Mandelbrot Jubilee

As the subtitle of this volume indicates, the conference out of which these proceedings arose was held (in 2002) in honor of the jubilee of Benoît Mandelbrot. Naturally, the curious reader may wonder what the word 'jubilee' is referring to here. In fact, the occasion for the meeting was the fiftieth anniversary (i.e., the jubilee) of Mandelbrot's thesis.[24]

Benoît Mandelbrot's Ph.D. dissertation [**Ma1**] was defended on December 16, 1952, at the Université de Paris in France and was entitled *Contribution à la Théorie Mathématique des Communications* [Contribution to the Mathematical Theory of Communications]. The Ph.D. committee was presided over by the Physics Nobel Laureate Louis de Broglie (whose own thesis had played a key role in the early developments of quantum mechanics). As was (and still is) frequently the case in

[23]Dr. Musgrave gave an after-dinner talk on this same subject after the banquet associated with the conference and AMS special session bearing the title of this volume in January 2002 (in San Diego).

[24]Interestingly and perhaps not coincidentally, the publication year (2004) and month (December) of this volume will also correspond pretty closely with the eightieth birthday of Benoît Mandelbrot. (He was born towards the end of November 1924.)

France, the thesis was not really directed by anyone in particular,[25] and hence there is no need to include here the names of the other members of the committee (or 'jury').[26]

Regarding Mandelbrot's thesis, Alfred Kastler (another French laureate of the Nobel Prize in Physics) once said that that the first half was about a subject that did not yet exist whereas the second half was about a subject which no longer existed. It is of interest to see how Mandelbrot himself describes it (in [**Ma12**]):

> *In 1952, I submitted a Ph.D. dissertation, unfinished and poorly presented, that largely determined the course of my life.*
>
> *Roughly speaking, the first half was about mathematical linguistics, the second, about statistical thermodynamics, the first half being a very exotic form of the second.*
>
> *Onlookers—even friendly ones—all warned me that the combination was by no means "natural". Not only did the first half concern a subject that didn't yet exist, but it was clear that my main goal was not to help linguistics become mathematical, but to explain a mysterious power-law relation called Zipf's law. George Kingsley Zipf (1902–1950) claimed that power laws characterize everything interesting in the social sciences, and provide them with an element of unity, in contrast to the physical sciences.* (...)

---

[25]It may be worth mentioning that among the early scientific influences of Benoît Mandelbrot, one should place at the very top his own uncle, Szolem Mandelbrojt (1899–1983), who was a Professor at the Collège de France in Paris, where he held the chair of Mathematics and Mechanics previously occupied by Jacques Hadamard. (Later on, beginning in 1973, Szolem Mandelbrojt also occupied at the French Académie des Sciences a chair once held by Henri Poincaré and then Jacques Hadamard.) Regarding his uncle—who early on pointed out to him as a worthy subject of study the then somewhat forgotten work of Gaston Julia and Pierre Fatou on the iteration of rational functions (see the end of Section 2) but whose other mathematical tastes and inclinations were very different from his own—, Mandelbrot writes in [**Ma12**]: "*No one influenced my scientific life nearly as much as this uncle did.*" Another key early scientific mentor of Benoît Mandelbrot was the French probabilist Paul Lévy (briefly mentioned in several places in Section 2 above), who was his professor at the École Polytechnique in Paris and had made wonderful contributions to probability theory and analysis but was not considered to be sufficiently rigorous by many of the other contemporary French mathematicians. (One should keep in mind that the Bourbaki school was then overwhelmingly dominant in French mathematics. Indeed, in a recent phone conversation with the present writer, Benoît Mandelbrot described himself as an exile driven away from France by the Bourbaki school, much like the French philosopher, mathematician, and scientist, René Descartes (1596–1650) had once been driven away from Paris to the Netherlands because of his 'heretic' philosophical views.)

[26]Later important scientific influences of Benoît Mandelbrot include Norbert Wiener and John von Neumann. Regarding this last point, Mandelbrot writes in [**Ma12**]: "In launching this unorthodox career [shortly after the defense of my thesis], I was greatly influenced by the examples of *Game Theory and Economic Behaviour* by John von Neumann (1903–1957) and *Cybernetics* by Norbert Wiener (1894–1964). Each seemed to be a bold attempt to put together and develop an appropriate new mathematical approach to very old and very concrete problems that overlapped several disciplines. I spent 1953-1954 at the Institute for Advanced Study in Princeton as the last post-doc sponsored by von Neumann; a Rockefeller Foundation grant was arranged by Warren Weaver. At one point in the sixties, Weaver revealed having been asked by "Johnny" (then dying of cancer) to keep an eye on me, because my chosen lifestyle was dangerous and I may need help."

> *Worse even than the first half of my thesis, the second half dabbled with a subject that was viewed as no longer part of active physics. This was reasonably true in 1952, but not for long. In the late 1960s and early 1970s, the study of critical phenomena ushered a brilliant period in the history of statistical physics. What did it bring? It brought renormalization, fixed points, and a contingent of new scaling laws that became better understood in that context than anything I used to know.* (...)
>
> *All in all, the details of my 1952 dissertation were never important. My explanation of Zipf's law keeps being rediscovered only to sink back into oblivion.*
>
> *To the contrary, the broad central argument of my thesis—the appreciation of the significance of the power laws and their explanation (soon broadened by renormalization and fixed points) —proved to have very "strong legs". Also, much of my work outside of mainstream physics has been directed by a "generalized thermodynamics". The great J. W. Gibbs has been criticized for fathering a theory that went beyond gases, to include "assemblies of sewing machines" (I quote from memory). The thermodynamics of roughness may have gone even beyond "assemblies of sewing machines".*

We now further examine Mandelbrot's motivations for focusing on this then somewhat obscure area of science that later came to be known as 'fractal geometry' and also overlapped so many a priori unconnected disciplines (from telecommunication theory to geology and astronomy, in passing by thermodynamics and statistical physics, anatomy, economics, and of course, various fields of mathematics). For this purpose, it may be best to quote one last time from Mandelbrot's recent autobiographical essay[27] entitled *A maverick's apprenticeship* and to be included in a forthcoming volume about the recipients of the Wolf Prize (which he was awarded for Physics in 1993):[28]

> *My whole career became one long, ardent pursuit of the concept of roughness. The roughness of clusters in the physics of disorder, of turbulent flows, of exotic noises, of chaotic dynamical systems, of the distribution of galaxies, of coastlines, of stock-prize charts, and of mathematical constructions. These have typified the topics I studied.*
>
> *The field of fractal and multifractal geometry, which I have been credited as founding, is the underpinning of a new, emerging theory of roughness. "Fractal", a word I coined for a new concept whose roots are traceable over centuries and millennia, denotes*

[27] I wish to thank Professor Mandelbrot for sending me an advance copy of this essay and for allowing me to include extended excerpts from this text, for which he has retained the copyright.

[28] The citation for this prize was the following: "By recognizing the widespread occurrence of fractals and developing mathematical tools for describing them, he has changed our view of nature."

geometric shapes that break into parts, each a small-scale model of the whole. (...)

To start towards a comprehensive and harmonizing approach to a sensory input that had long defied rational study, a new geometry turned out to be necessary.

How did I come upon one? Answer: Not in one Eurêka! flash, but step by step, in a painful process that continues to reveal new horizons. I had chosen to be an "outsider" to most of the fields where I worked. Conceiving fractal geometry, then developing it in many directions, required contributions to a broad range of topics. Those included several branches of physics or pure mathematics, and several other sciences.

To what established discipline does my work belong? To none exclusively—and to many. One crucial unifying thread is that I am an avid student of thermodynamics and since 1951 have generalized its key ideas to an increasingly wide variety of unconventional complex systems. These ideas were not used as a metaphor but as the main basis for technical work. That is why every step of my work involves the word "macroscopic".

Another unifying thread: a constant pursuit of scale-invariances and symmetries. Further threads: power-law relations rule all the systems I studied and a strong geometric mode of thought eventually led to fractals. In pure mathematics, my main contribution has not been to provide proofs, but to ask new questions—usually very hard ones—suggested by physics and pictures. (...)

In due time, persistence in the study of roughness made me encounter increasing depths of wild complexity and conclude that the world is not fundamentally mild and simple. In due time, the intellectual landscape that I chose to visit as scientist turned out to have been previously visited by my uncle and his friends. The hard messiness found in his mathematics may well reflect the irreducible messiness of the scientific frontier where I have chosen to work.

As he himself gladly recognizes, both in private conversations and in written texts such as the preceding one, Mandelbrot is far from being a traditional mathematician, who formulates precise theorems and supplies complete proofs for them. Instead, he works in many different fields (often quite distant from pure mathematics) and establishes previously unknown connections between them. Then, after a long and patient research or maturation, he proposes new and typically very challenging conjectures, which other mathematicians tackle and in some cases, eventually solve. (Examples of such conjectures are discussed in the second chapter of Part 2, for instance; see the latter part of Section 2 above.)[29]

[29]Felix Browder (personal communication, April 2003), on the occasion of a lecture, for example, likes to introduce Benoît Mandelbrot as a "phenomenological mathematician". This sheds an interesting light on this matter.

It is fair to say that Mandelbrot's relations with certain mathematicians have often been (and probably will continue to be) uneasy. This is due in part to a common (and unfortunate) misunderstanding about his own appreciation of the importance and the role of proofs in mathematics. Based on this writer's frequent conversations with him on this very subject, it seems accurate to write that Mandelbrot considers that the formulation of interesting problems or conjectures on the one hand, and the formulation and the proof of specific theorems on the other hand, are both a vital (and complementary) part of the activity of mathematicians (possibly, of different types of mathematicians or 'mathematical scientists').

We leave to future historians of science and of mathematics, in particular, the task of exploring this theme in more depth, but there is no denying that Mandelbrot has admirably played (and remarkably, continues to play) the role of the poet, as it was beautifully described by Arthur Rimbaud: he has "raised a part of the veil" and gave us all to see previously unexplored territories. For this wonderful contribution alone, we should be grateful to him because our mathematical and scientific horizons have been significantly broadened as a result.

Just before the main body of Part 1 below, the interested reader will find futher discussion of Mandelbrot's scientific work and personality in a series of thirteen short personal essays or 'tributes' by a variety of contributors to this volume, speakers in one of the aforementioned meetings, as well as by several other mathematicians or scientists.[30] These authors were asked by the editors to give (in a few pages) their own view of some of Mandelbrot's mathematical or scientific contributions and to explain how they influenced their own work, and/or to evoke personal anecdotes or souvenirs regarding their own relationship with Benoît Mandelbrot.[31] We hope that these will be enjoyable and reveal new facets previously unknown to the reader.

Mandelbrot's scientific contributions are widely known by now, ever since the publication in 1983 of his groundbreaking essay *The Fractal Geometry of Nature* [**Ma7**], preceded by the publication in French of his 1977 book *Les Objets Fractals: Forme, hasard et dimension* [**Ma5**]. They are also summarized and commented upon by Mandelbrot himself in his selected works, part of which have appeared in [**Ma8**]–[**Ma11**] on a variety of subjects, 'fractals and power laws in finance' [**Ma8**], 'multifractals' [**Ma9**], 'Gaussian self-affinity and its applications' [**Ma10**], and 'complex dynamics, including the Mandelbrot set' [**Ma11**]. We will therefore not say any more about it here, especially in view of the tributes following this introduction.

A Personal Note. It may be appropriate to close this paper by some brief personal comments and recollections. Like many others, I first came in contact with Mandelbrot's universe by reading (as soon as 1983, I believe) his classic book *The Fractal Geometry of Nature* [**Ma7**]. (Somehow, I first read his English book.) I was mesmerized by this truly unusual book or 'essay', which did not resemble anything

[30]In particular, Michael Berry was one of the speakers at the AMS special session on "Fractal Geometry and Applications" and was also the Gibbs Lecturer at the same Annual Meeting of the American Mathematical Society in San Diego. Furthermore, Jean-Pierre Kahane and David Mumford were unable to come to the meeting, due to prior commitments, but were still able to write a tribute for this volume.

[31]This was done without his knowledge, of course.

I had read at the time (or have read since then). In fact, I was so shaken by it that I went to seek the opinion and advice of Mark Kac, who was then my mentor, friend, and senior colleague at the University of Southern California in Los Angeles (where I was an assistant professor). I asked him what he thought of this 'essay', which was so different from all the mathematics books written in the Bourbaki style I had been used to reading during my studies in Paris.

I told Mark Kac that I was fascinated by this subject but was wondering whether it could be the object of serious mathematical investigations since there did not seem to be any real theorems or precise definitions stated in [**Ma7**], only beautiful pictures and a number of mind-boggling observations and statements or conjectures apparently directly relevant to the study of nature.[32] Mark Kac, who knew me well (and to whom I had given a collection of poems I had written a few years earlier for my future wife), answered me (talking about Mandelbrot): "*He is a poet, like you.*" He then went on to strongly encourage me to further explore this subject, and not to formalize myself about the apparent lack of rigorous theorems in this area. Not long afterwards, as I began to get more and more interested in the question *Can one hear the shape of a fractal drum*?, for domains with progressively rougher boundaries, he also warmly encouraged me to continue on this path. These were two of his last (but long-lasting) gifts to me.

My first direct encounter with Benoît Mandelbrot took place during a conference held at the École Polytechnique in the mid-to-late 1980's to commemorate the centennial of the birth of Paul Lévy (1886–1971).[33] I had then obtained my first results on the theory of vibrations of 'fractal drums' and the Weyl–Berry conjecture [**Ber**] (later developed from several points of view with many wonderful collaborators in, e.g.,[**HaLa**], [**HeLa**], [**KiLa1**], [**KiLa2**], [**La1**]–[**La5**], [**LaFl**], [**LaMai**], [**LaNRG**], [**LaPa**], [**LaPea**], [**LaPo1**], [**LaPo2**], [**La-vF**], as well as by a number of other researchers (see, e.g., the relevant references in [**La2**]–[**La4**] and [**La-vF**]). Mandelbrot listened attentively to my explanations and seemed very interested. He clearly knew the background of the problem, both in the physics and mathematics literature. (He was aware, in particular, of references [**Ber**] and [**BrCa**].) His excellent knowledge of the literature and of the main researchers in many different fields touching upon fractal geometry (among a variety of other topics) is something that I was able to confirm over and over again since then.

Some time later, Benoît Mandelbrot (along with Ronald Coifman) invited me as a visiting professor to teach and do research at Yale University during the academic year 1990–1991. We have had many wide-ranging mathematical conversations during that period, some of which were about his own approach to the notion of 'lacunarity' and 'texture', which he was in the process of revisiting. We also started discussing the outline of an undergraduate course on fractal geometry I had

[32]The 'culture shock' I experienced in reading and mulling over [**Ma7**] was akin to the one I had experienced a little earlier during my first few months in Berkeley, freshly arrived from Paris, while participating in the extraordinary 'Gauge Theory and Mathematical Physics Seminar' of Isadore Singer, attended by mathematicians and physicists of every possible background and with varying tastes and appreciation for rigor. There, when presenting new concepts or fields, contrary to the Bourbaki dogma, a series of interesting examples came first, well before any precisely formulated definitions were provided.

[33]Prior to that, in the early 1980's, I had already attended a lecture by Mandelbrot during a regional SIAM meeting in Los Angeles, but had not talked to him on that occasion.

just been asked to create at the University of California, Riverside, where I was about to move. Benoît gave me a copy of the lecture notes of a course which he had taught at Harvard on this theme.

As one of the tributes (by Michael Frame) will make it plain, Mandelbrot's passion for his subject remarkably extends to help developing teaching material and curriculum at all levels, from the high school and college level, all the way up to the graduate and research level. (See, e.g., the recent collective volume [**FrMa**] for one illustration of this point.) On my side (as is explained in [**La7**], from [**FrMa**]), I ended up creating and teaching many such courses and related honors seminars, which I have enjoyed enormously. It was always a pleasure to tell Benoît about it and to see his face glow, as a result.

Over the years, we have met on numerous occasions, be it in New York or Connecticut, at common conferences around the world, at Cambridge University where we were both participating for several months in 1999 in a program on fractal geometry at the Newton Institute, or at my own university in California (where he delivered the Regents Lectures of the University of California, Riverside, a few years ago). We have had a large number of very interesting conversations, both directly and over the phone (often weekly), for hours at a time, ranging from music, world history, politics, to all kinds of topics in science and mathematics, as well as more personal matters. He would always make a point when calling me at home or in my office to ask about my children and my wife and would remember every detail about them. (He has witnessed the first few steps of my daughter, Julie, in his office at Yale University in October 1990. He immediately noticed that she had moved her right foot first and declared that she would be right-handed, which she is.) As several of the tributes will confirm, he is genuinely interested in the well-being of his collaborators[34] and friends.[35]

In addition to our common passion for mathematics and physics, Benoît and I (or my father) share several cultural traits and geographical origins. We are both French and have received all or most of our education in France (including in the same advanced mathematics classes at the Lycée Louis Le Grand in Paris). His family was Jewish from Poland, and so was my father's family. In fact, Benoît was born (on November 20, 1924) in the capital, Warsaw, while my father was born (on December 15 of that same year) in Byalistock (in the Northeast of Poland, close to Byelorussia), and one of our recent ancestors was the grand rabbi of these respective cities. Both my father and he emigrated from Poland to France during their

---

[34]No later than a few months ago, during the preparation of this volume, I have witnessed a striking example of such concern, towards one of his collaborators. Discretion requires that I do not say more about it here.

[35]Accordingly, one of Benoît's recent gifts to me was to encourage me enthusiastically but forcefully to present in book form (rather than in a very long paper in a collective volume, as was then my intention) the theory (or program) now developed in the forthcoming book (or 'essay') [**La5**]. Almost in its final form, the latter is presently approaching 450 pages. Hence, Benoît's foresight and kind insistence were completely justified. Fittingly, even though I had been thinking about it for many years (at least, since 1996 or 1997) and had already written nearly 200 pages about it, this theory was presented publicly for the first time in the lecture I gave at the Mandelbrot Jubilee in San Diego in January 2002.

childhood, and essentially all the rest of their family disappeared during the Holocaust, back in Poland. Furthermore, Benoît's wife, Aliette, was born in Byalistock, where her own family was from.[36]

It would be impossible to close these personal remarks without mentioning the indispensable role played by Benoît's wife, Aliette, whose warm presence and support enlivens every meeting and social occasion. Benoît and Aliette's personalities and interests complement each other to such an extent that it is difficult to think of one without the other.

In conclusion, it is a pleasure for me to dedicate this paper to Benoît Mandelbrot, on the occasion of the Jubilee of his thesis and of his forthcoming eightieth birthday.

Acknowledgements. I wish to thank my Ph.D. student, Erin Pearse, for helpful feedback and comments on a preliminary version of this paper.

References

[AhFe] A. Aharony and J. Feder, *Fractals in Physics*: *Essays in honour of Benoît Mandelbrot* (65th birthday celebration), North-Holland, Amsterdam, Physica D **38**, 1989.

[Al] D. S. Alexander, *A History of Complex Dynamics*: *from Schröder to Fatou and Julia*, Vieweg, Braunschweig, 1994.

[Ba] L. J. B. A. Bachelier, *Théorie de la Spéculation*, Ann. Sci. École Norm. Sup. (3) **17** (1900), 21–86. (English translation available in: *The Random Character of Stock Market Prices* (P. H. Cootner, ed.), MIT Press, Cambridge, Mass., 1964, pp. 17–78.)

[Bar] M. Barlow, *Diffusions on fractals*, in: Lectures on Probability Theory and Statistics, École d'Été de Probabilités de Saint-Flour XXV, Lecture Notes in Math., Vol. 1690, Springer-Verlag, Berlin, 1998, pp. 1–121.

[BenA] G. Ben Arous, *Géométrie de la courbe brownienne plane*, in: Séminaire Bourbaki, 43ème année, 1990-91, No. 730, Soc. Math. France, Astérisque **201** (1991), 7–42.

[Ber] M. V. Berry, *Distribution of modes in fractal resonators*, in: Structural Stability in Physics (W. Güttinger and H. Eikemeier, eds.), Springer-Verlag, Berlin, 1979, pp. 51–53.

[Bes1] A. S. Besicovitch, *On linear sets of fractional dimensions*, Math. Ann. **101** (1929), 161–193.

[Bes2] A. S. Besicovitch, *On sets of fractional dimensions* II: *On the sum of digits of real numbers represented in the dyadic system*, Math. Annalen **110** (1934), 321–330. (Reprinted with annotations in [Ed, pp. 145–159].)

[Bes3] A. S. Besicovitch, *On sets of fractional dimensions* III: *Sets of points of non-differentiability of absolutely continuous functions and of divergence of Fejér sums*, Math. Annalen **110** (1934), 331–335.

[Bes4] A. S. Besicovitch, *Sets of fractional dimensions* IV: *On rational approximation to real numbers*, J. London Math. Soc. **9** (1934), 126–131. (Reprinted with annotations in [Ed, pp. 161–168].)

[BesTay] A. S. Besicovitch and S. J. Taylor, *On the complementary intervals of a linear closed set of zero Lebesgue measure*, J. London Math. Soc. **29** (1954), 449–459. (Reprinted with annotations in [Ed, pp. 269-282].)

[BesUr] A. S. Besicovitch and H. D. Ursell, *Sets of fractional dimensions* V: *On dimensional numbers of some continuous curves*, J. London Math. Soc. **12** (1937), 18–25. (Reprinted with annotations in [Ed, pp. 171–179].)

[Bol] B. Bolzano, *Funktionenlehre*, Herausgegeben und mit Anmerkungen versehen von K. Rychlik, Prague, 1930.

[36] Regrettably, at least so far, my father has never had the occasion to meet either Benoît or Aliette.

[Bou] G. Bouligand, *Ensembles impropres et nombre dimensionnel*, Bull. Sci. Math. (2) **52** (1928), 320–344 & 361–376. (English translation of excerpts reprinted with annotations in [Ed, pp. 119-131], under the title *Improper sets and dimension numbers*.)

[BrCa] J. Brossard and R. Carmona, *Can one hear the dimension of a fractal*? Commun. Math. Phys, **104** (1986), 103–122.

[Can] G. Cantor, *On the power of perfect sets of points*, English translation (by I. Vardi) of the French original (*De la puissance des ensembles parfaits de points*, Acta Math. **4** (1884), 381–392), reprinted with annotations in [Ed, pp. 11–23].

[Car] C. Carathéodory, *On the linear measure of point sets—a generalization of the concept of length*, English translation (by B. Sawhill) of the German original (*Über das lineare Mass von Punktmengen-eine Verallgemeinerung des Längenbgriffs*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1914), 404–426), reprinted with annotations in [Ed, pp. 47–73].

[Ce] E. Cesàro, *Remarques sur la courbe de von Koch*, Att. Accad. Sci. Fis. Mat. Napoli **12** (1905), 1–12.

[Co] D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, 1980.

[Col] J. Collins, *Renormalization*: *An introduction to renormalization, the renormalization group, and the operator product expansion*, Cambridge Univ. Press, Cambridge, 1984.

[Da] J. W. Dauben, *Georg Cantor*: *His mathematics and philosophy of the infinite*, Harvard Univ. Press, Cambridge, Mass., 1979.

[dR] G. de Rham, *On some curves defined by functional equations*, English translation (by I. Vardi) of the French original (*Sur quelques courbes définies par des équations fonctionnelles*, Rend. Sem. Mat. Univ. Politec. Torino **16** (1957), 101–113), reprinted with annotations in [Ed, pp. 285–297].

[Di1] J. P. G. Lejeune Dirichlet, *Sur la convergence des séries trigonométriques qui servent a représenter une fonction arbitraire entre des limites données*, J. Reine Angew. Math. (Crelle's Journal) **4** (1828), 157–169. (Reprinted in [Di2,pp.117–132.)

[Di2] J. P. G. Lejeune Dirichlet, *Werke*, Vol. 1, Reimer, Berlin, 1889.

[Do] J. L. Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer-Verlag, New York, 1984.

[DuKw] B. Duplantier and K.-H. Kwon, *Conformal invariance and intersection of random walks*, Phys. Rev. Lett. **61** (1988), 2514–2517.

[Ed] G. A. Edgar (ed.), *Classics on Fractals*, Addison-Wesley, Reading, Mass., 1993.

[Ei] A. Einstein, *Investigations on the Theory of the Brownian Movement*, Dover, New York, 1956. (English translation of Einstein's five papers on Brownian motion; edited with notes by R. Fürth.)

[EvPeiVo] C. J. G. Evertsz, H.-O. Peitgen and R. F. Voss (eds.), *Fractal Geometry and Analysis*: *The Mandelbrot Festschrift*, World Scientific, Singapore, 1996.

[Fa1] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.

[Fa2] K. J. Falconer, *Fractal Geometry*: *Mathematical foundations and applications*, Wiley, London, 1990.

[Fa3] K. J. Falconer, *Techniques in Fractal Geometry*, Wiley, Chichester, 1997.

[Fat1] P. Fatou, *Sur les équations fonctionnelles*, Bull. Soc. Math. France **47** (1919), 161–271; **48** (1920), 33–94 & 208–314.

[Fat2] P. Fatou, *Sur les équations fonctionnelles*, Gauthiers-Villars, Paris, 1922. (Reprinting of the three papers in [Fat1].)

[Fe] J. Feder, *Fractals*, Plenum Press, New York, 1988.

[Fed] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[Fo] J. Fourier, *Théorie Analytique de la Chaleur*, Mémoires de l'Académie des Sciences, Paris, 1822. (Fourier's prize-winning essay of 1812.)

[FrMa] M. L. Frame and B. B. Mandelbrot, *Fractals, Graphics, & Mathematics Education*, Math. Assoc. America, Washington, D. C., 2002.

[GrKa] L. Graham and J.-M. Kantor, *Name worshippers*: *Religion, Russian and French Mathematics, 1900–1930*, Preprint, 2004.

[HaLa] B. Hambly and M. L. Lapidus, *Random fractal strings*: *Their zeta functions, complex dimensions and spectral asymptotics*, Trans. Amer. Math. Soc. (2004), in press.

[Har] G. H. Hardy, *Weierstrass' non-differentiable function*, Trans. Amer. Math. Soc. **17** (1916), 301–325.

[Hau] F. Hausdorff, *Dimension and outer measure*, English translation (by B. Sawhill) of the German original (*Dimension und äusseres Mass*, Math. Annalen **79** (1918), 157–179), reprinted with annotations in [Ed, pp. 75–100].

[Haw] T. Hawkins, *Lebesgue's Theory of Integration*: *Its origins and developments*, Amer. Math. Soc., Providence, R. I., 2001.

[HeLa] C. Q. He and M. L. Lapidus, *Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function*, Memoirs Amer. Math. Soc. No. 608, **127** (1997), 1–97.

[HerSt] C. Hermite and T. J. Stieltjes, *Correspondance*, Vol. II, Gauthiers-Villars, Paris, 1905.

[Hu] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[ItZu] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw Hill, New York, 1980.

[JoLa] G. W. Johnson and M. L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus*, Oxford Mathematical Monographs, Oxford Univ. Press, Oxford, 2000. (Corrected reprinting and paperback edition, 2002.)

[Ju1] G. Julia, *Mémoire sur l'itération des fonctions rationnelles*, J. Math. Pure Appl. **8** (1918), 47–245. (Reprinted (with several related papers) in [Ju2,pp.171–319].)

[Ju2] G. Julia, *Oeuvres de Gaston Julia*, Gauthiers-Villars, 1968.

[Ki] J. Kigami, *Analysis on Fractals*, Cambridge Univ. Press, Cambridge, 2001.

[KiLa1] J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Commun. Math. Phys. **158** (1993), 93–125.

[KiLa2] J. Kigami and M. L. Lapidus, *Self-similarity of volume measures for Laplacians on p.c.f. self-similar fractals*, Commun. Math. Phys. **217** (2001), 165–180.

[Kol] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Chelsea, New York, 1956. (Originally published as *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.)

[KolTi] A. N. Kolmogorov and V. M. Tihomirov, *ϵ-entropy and ϵ-capacity of sets in functional spaces*, Ouspeki Mathem. **14** (1959), 3–86. English translation in Amer. Math. Soc. Transl. Ser. 2, **17** (1961), 277–364. (Excerpts reprinted in [Ed, pp. 299–339].)

[La1] M. L. Lapidus, *Fractal drums, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture*, Trans. Amer. Math. Soc. **325** (1991), 1–55.

[La2] M. L. Lapidus, *Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl–Berry conjecture*, in: Ordinary and Partial Differential Equations (B. D. Sleeman and R. J. Jarvis, eds.), Vol. IV, Proc. Twelfth Internat. Conf. (Dundee, Scotland, UK, June 1992), Pitman Research Notes in Math. Series, Vol. 289, Longman and Technical, London, 1993, pp. 126–209.

[La3] M. L. Lapidus, *Fractals and vibrations*: *Can you hear the shape of a fractal drum*? Fractals **3**, No. 4 (1995), 725–726. (Special issue in honor of Benoît B. Mandelbrot's 70th birthday. Reprinted in [EvPeiVo,pp.321–332].)

[La4] M. L. Lapidus, *Towards a noncommutative fractal geometry*? *Laplacians and volume measures on fractals*, Contemporary Mathematics, Amer. Math. Soc. **208** (1997), 211–252.

[La5] M. L. Lapidus, *In Search of the Riemann Zeros*: *Strings, fractal membranes, and noncommutative spacetimes*, forthcoming research book in preparation; Preprint, May 2004, 430+(xiii) pp.

[La6] M. L. Lapidus, *The Sierpinski gasket*, in: Kluwer Encyclopedia of Mathematics, Suppl. Vol. III, Kluwer Academic Publ., 2002, pp. 364–368.

[La7] M. L. Lapidus, *Creating and teaching undergraduate courses and seminars in fractal geometry*: *A personal experience*, in [FrMa] above: Fractals, Graphics, and Mathematics Education (M. L. Frame and B. B. Mandelbrot, eds.), Math. Assoc. America, Washington, D. C., 2002, pp. 111–116.

[LaFl] M. L. Lapidus and J. Fleckinger-Pellé, *Tambour fractal*: *vers une résolution de la conjecture de Weyl–Berry pour les valeurs propres du laplacien*, C. R. Acad. Sci. Sér. I Math. **306** (1988), 171–175.

[LaMai] M. L. Lapidus and H. Maier, *The Riemann hypothesis and inverse spectral problems for fractal strings*, J. London Math. Soc. (2) **52** (1995), 15–34.

[LaNRG] M. L. Lapidus, J. W. Neuberger, R. J. Renka and C. A. Griffith, *Snowflake harmonics and computer graphics: Numerical computation of spectra on fractal domains*, Internat. J. Bifurcations & Chaos **6** (1996), 1185–1210.

[LaPa] M. L. Lapidus and M. M. H. Pang, *Eigenfunctions of the Koch snowflake drum*, Commun. Math. Phys. **172** (1993), 359–376.

[LaPea] M. L. Lapidus and E. P. J. Pearse, *A tube formula for the Koch snowflake curve, with applications to complex dimensions*, Preprint, 2004.

[LaPo1] M. L. Lapidus and C. Pomerance, *The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums*, Proc. London Math. Soc. (3) **66** (1993), 41–69.

[LaPo2] M. L. Lapidus and C. Pomerance, *Counterexamples to the modified Weyl–Berry conjecture on fractal drums*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 167–178.

[La-vF] M. L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry and Number Theory: Complex dimensions of fractal strings and zeros of zeta functions*, Birkhäuser, Boston, 2000. (Second revised and expanded edition to appear in 2005.)

[Law] G. F. Lawler, *Conformal invariance, universality, and the dimension of the Brownian frontier*, in: Proc. Internat. Congress Math. (Beijing, China, 2002), Vol. III, 2002, Higher Education Press, Beijing, pp. 63–72.

[LawSW1] G. F. Lawler, O. Schramm and W. Werner, *The dimension of the planar Brownian frontier is* $\frac{4}{3}$, Math. Res. Lett. **8** (2001), 13–23.

[LawSW2] G. F. Lawler, O. Schramm and W. Werner, *Values of Brownian intersection exponents.* I: *Half-plane exponents*, Acta Math. **187** (2001), 237–273; II: *Plane exponents*, Acta Math. **187** (2001), 275–308; III: *Two-sided exponents*, Ann. Inst. H. Poincaré Probab. Statist. **38** (2002), 109–123.

[Leb1] H. Lebesgue, *Sur l'intégration des fonctions discontinues*, Ann. Sci École Norm. Sup. (3) **27** (1910), 361–450.

[Leb2] H. Lebesgue, *Measure and the Integral*, (K. O. May, ed.), Holden Day, Inc., San Francisco, 1966. (See also, e.g., H. Lebesgue, *Leçons sur l'Intégration et la Recherche des Fonctions Primitives*, Paris, 1904.)

[Leb3] H. Lebesgue, *Oeuvres Scientifiques*, (F. Châtelet and G. Choquet, eds.), Vols. I–V, Institut de Mathématique, Geneva, 1972–1973.

[Lei] T. Lei (ed.), *The Mandelbrot Set, Theme and Variations*, London Math. Soc. Lecture Note Ser., Vol. 274, Cambridge Univ. Press, Cambridge, 2000.

[Lev1] P. Lévy, *Stochastiques et Mouvement Brownien*, Editions Jacques Gabai, revised and enlarged edition (from the 1965 Gauthiers-Villars second edition), Paris, 1992.

[Lev2] P. Lévy, *Plane or space curves and surfaces consisting of parts similar to the whole*, English translation (by I. Vardi) of the French original (*Les courbes planes ou gauches et les surfaces composées de parties semblables au tout*, Journal de l'École Polytechnique (1938), 227–247 & 249–291), reprinted with annotations in [Ed, pp. 181–239].

[LoMNW] G. A. Losa, D. Merlini, T. F. Nonnemacher and E. R. Weibel (eds.), *Fractals in Biology and Medicine*, Vols. I–III, Birkhäuser, Basel, 1994, 1998 and 2002.

[Ma1] B. B. Mandelbrot, *Contribution à la Théorie Mathématique des Communications*, Thèse de Doctorat ès Sciences, Université de Paris, France, 1953.

[Ma2] B. B. Mandelbrot, *How long is the coast of Britain? Statistical self-similarity and fractional dimension*, Science **156** (1967), 636–639. (Reprinted in [Ed, pp. 351–358].)

[Ma3] B. B. Mandelbrot, *Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée*, C. R. Acad. Sci. Paris Sér. A **278** (1973), 289–292 & 355–358.

[Ma4] B. B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergences of high moments and dimensions of the carrier*, J. Fluid Mech. **62** (1974), 331–358.

[Ma5] B. B. Mandelbrot, *Les Objets Fractals: Forme, hasard et dimension*, Flammarion, Paris, 1975.

[Ma6] B. B. Mandelbrot, *Fractal aspects of the iteration* $z \to \lambda z(1-z)$ *for complex* λ *and* z, Ann. New York Acad. Sci. **57** (1980), 249–259.

[Ma7] B. B. Mandelbrot, *The Fractal Geometry of Nature*, revised and enlarged edition, W. H. Freeman, New York, 1983.

[Ma8] B. B. Mandelbrot, *Selecta, Volume E: Fractals and Scaling in Finance: Discontinuity, concentration, risk*, Springer-Verlag, NewYork, 1997.

[Ma9] B. B. Mandelbrot, *Selecta, Volume N: Multifractals and* $\frac{1}{f}$ *Noise: Wild self-affinity in physics*, Springer-Verlag, New York, 1999.

[Ma10] B. B. Mandelbrot, *Selecta, Volume H: Gaussian Self-affinity and Fractals: $R/S, 1/f$, globality, reliefs and rivers*, Springer-Verlag, New York, 2002.

[Ma11] B. B. Mandelbrot, *Selecta, Volume C: Fractals and Chaos: The Mandelbrot set and beyond*, Springer-Verlag, New York, 2004.

[Ma12] B. B. Mandelbrot, *A maverick's apprenticeships*, Preprint, 2004; text to appear in *The Wolf Prize for Physics*, D. J. Thoules (ed.), World Scientific, Singapore, 2005.

[Mat] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge Univ. Press, Cambridge, 1995.

[Matt] R. D. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem*, second edition, McGraw Hill, New York, 1976.

[Me] K. Menger, *General spaces and Cartesian spaces*, English translation (by G. A. Edgard) of the German original (*Allgemeine Räume und Cartesische Räume*, Proc. Sciences Section, Koniklijke Akademie van Wetenschappen te Amsterdam **29** (1926), 476–482 and 1125–1128), reprinted with annotations in [Ed, pp. 103–116].

[Mo] P. A. P. Moran, *On plane sets of fractional dimension*, Proc. London Math. Soc. **51** (1949), 415–423.

[Ne] E. Nelson, *Dynamical Theories of Brownian Motion*, Math. Notes, Princeton Univ. Press, Princeton, 1967.

[Pe] G. Peano, *Sur une courbe, qui remplit toute une aire plane*, Math. Annalen **36** (1890), 157–160.

[Pea] E. P. J. Pearse, *Universality of the Sierpinski carpet*, Undergraduate Honors Thesis, Mathematics, Honors Program, Univ. of California, Riverside, 1998.

[PeiJS] H.-O. Peitgen, H. Jürgens and D. Saupe, *Chaos and Fractals: New frontiers of science*, Springer-Verlag, New York, 1992.

[PeiR] H.-O. Peitgen and P. H. Richter, *The Beauty of Fractals: Images of complex dynamical systems*, Springer-Verlag, New York, 1986.

[Per] J. Perrin, *Les Atomes*, New edition (of the 1913 original edition), Flammarion, Paris, 1991.

[Pes] Y. B. Pesin, *Dimension Theory in Dynamical Systems: Contemporary view and applications*, Chicago Lectures in Mathematics, Univ. of Chicago Press, Chicago, 1997.

[Polc] J. Polchinski, *String Theory*, Vols. I & II, Cambridge Univ. Press, Cambridge, 1998.

[Poly] A. M. Polyakov, *Gauge Fields and Strings*, Harwood Academic, Newark, 1987.

[Pom] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, New York, 1992.

[PonSch] L. Pontrjagin and L. Schnirelmann, *On a metric property of dimension*, English translation (by I. Vardi) of the French original (*Sur une propriété métrique de la dimension*, Ann. of Math. **33** (1932), 156–162), reprinted with annotations in [Ed, pp. 133–142].

[Ri] B. H. Riemann, Unpublished lecture, 1861 (or earlier). (Referred to by Karl Weierstrass in the introduction to his paper [We].)

[Ro] C. A. Rogers, *Hausdorff Measures*, Cambridge Univ. Press, Cambridge, 1970.

[Sc] M. R. Schroeder, *Fractals, Chaos, Power Laws: Minutes from an infinite paradise*, W. H. Freeman, New York, 1991.

[Si] W. Sierpinski, *Sur une courbe cantorienne et continue qui contient une image biunivoque et continue de toute courbe donnée*, C. R. Acad. Sci. Paris **162** (1916), 629–632.

[Sm] S. Smirnov, *Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 239–244. (Also, Preprint of the long version of this paper with the same short title.)

[Smi] H. J. S. Smith, *Collected Mathematical Papers*, 1894; reprinted by Chelsea Publ.

[Smo] M. Smoluchowski, *Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen*, Ann. Phys. (4) **21** (1906), 756–780.

[St] R. S. Strichartz, *Analysis on fractals*, Notices Amer. Math. Soc. **46** (1999), 1199–1208.

[Ta] T. Tagaki, Proc. Phys. Math. Soc. Japan Ser. II **1** (1903), 176.

[Tay] S. J. Taylor, *Sample path properties of processes with stationary independent increments*, in: Stochastic Analysis (a tribute to the memory of Rollo Davidson), Wiley, London, 1973, pp. 387–414.

[Tr1] C. Tricot, *Two definitions of fractional dimension*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 57–74.

[Tr2] C. Tricot, *Curves and Fractal Dimensions*, English translation (of the 1993 French edition), Springer-Verlag, New York, 1995.

[vK1] H. von Koch, *On a continuous curve without tangents constructible from elementary geometry*, English translation (by I. Vardi) of the French original (*Sur une courbe continue sans tangente obtenue par une construction géométrique élémentaire*, Ark. Mat., Astron. Fys. **1** (1904), 681–702), reprinted with annotations in [Ed, pp. 25–45].

[vK2] H. von Koch, *Une méthode géométrique élémentaire pour l'étude des courbes planes*, Acta Math. **30** (1906), 145–174.

[We] K. Weierstrass, *On continuous functions of a real argument that do not have a well-defined quotient*, English translation (by S. Sawhill) of the 1872 German original (*Über continuirliche Functionen eines reellen Arguments, die für keinen Werht des Letzteren einen bestimmten Differentialquotienten besitzen*, reprinted in *Karl Weierstrass*: *Mathematisches Werke*, Band II, Mayer and Müller, Berlin, 1895, pp. 71–74), reprinted with annotations in [Ed, pp. 3–9].

[Wi1] N. Wiener, *Differential space*, J. of Math. and Phys. (of MIT) **2** (1923), 131–174. (Reprinted in [Wi2, pp. 455-498], along with related papers.)

[Wi2] N. Wiener, *Norbert Wiener*: *Collected Works*, Vol. I, Wiley, New York, 1979.

[Wil] K. G. Wilson, *Renormalization group and critical phenomena*, I & II, Phys. Rev. B **4** (1971), 3174–3183 & 3184–3205.

[Wit] E. Witten, *Physics and geometry*, in: Proc. Internat. Congress Math. Berkeley, 1986 (A. M. Gleason, ed.), Vol. 1, Amer. Math. Soc., Providence, R. I., 1987, pp. 267–303.

MICHEL L. LAPIDUS, Managing Editor
University of California
Department of Mathematics, Surge Bldg.
Riverside, CA 92521-0135 USA
E-mail address: *lapidus@math.ucr.edu*